Published by Institute of Physics Publishing for SISSA

Received: *April 24, 2006*
Accepted: *August 24, 2006*
Published: *September 21, 2006*

Rank two quiver gauge theory, graded connections and noncommutative vortices

Olaf Lechtenfeld,[a] **Alexander D. Popov**[ab] **and Richard J. Szabo**[c]

[a] *Institut für Theoretische Physik, Leibniz Universität Hannover*
Appelstraße 2, 30167 Hannover, Germany
[b] *Bogoliubov Laboratory of Theoretical Physics, JINR*
141980 Dubna, Moscow Region, Russia
[3] *Department of Mathematics and Maxwell Institute for Mathematical Sciences*
Heriot-Watt University
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.
E-mail: lechtenf@itp.uni-hannover.de, popov@itp.uni-hannover.de, R.J.Szabo@ma.hw.ac.uk

Abstract: We consider equivariant dimensional reduction of Yang-Mills theory on Kähler manifolds of the form $M\times\mathbb{C}P^1\times\mathbb{C}P^1$. This induces a rank two quiver gauge theory on M which can be formulated as a Yang-Mills theory of graded connections on M. The reduction of the Yang-Mills equations on $M\times\mathbb{C}P^1\times\mathbb{C}P^1$ induces quiver gauge theory equations on M and quiver vortex equations in the BPS sector. When M is the noncommutative space $\mathbb{R}^{2n}_\theta$ both BPS and non-BPS solutions are obtained, and interpreted as states of D-branes. Using the graded connection formalism, we assign D0-brane charges in equivariant K-theory to the quiver vortex configurations. Some categorical properties of these quiver brane configurations are also described in terms of the corresponding quiver representations.

Keywords: Non-Commutative Geometry, Brane Dynamics in Gauge Theories, Solitons Monopoles and Instantons.

Contents

JHEP09(2006)054

1. Introduction and summary

It has become clear in recent years that a proper description of the nonperturbative vacuum in string theory will require detailed understanding of the properties of systems of both BPS and non-BPS brane configurations (see [1] for a recent review). The basic non-BPS system is the unstable brane-antibrane configuration which corresponds to a pair of vector bundles with a tachyon field mapping between them. The dynamics of this system can be cast as a Yang-Mills theory of superconnections [2]. In some instances the branes can be realized as instantons of gauge theory in the appropriate dimensionality [3]. Important examples

of this are noncommutative solitons and instantons which find their most natural physical interpretations in terms of D-branes [4]. This is related [5] to the fact that the charges of D-branes are classified by K-theory [6]. Reviews on noncommutative solitons and D-branes can be found in [7], while applications of BPS soliton solutions in noncommutative (supersymmetric) Yang-Mills theory to D-brane dynamics are given e.g. in [8].

One way to generate both stable and unstable states of D-branes is by placing them at singularities of orbifolds [9, 10]. Regular representation D-branes then decay into irreducible representation fractional branes under the action of the discrete orbifold group. The low-energy dynamics of the D-brane decay is succinctly described by a quiver gauge theory. Resolving orbifold singularities by non-contractible cycles blows up the fractional D-branes into higher dimensional branes wrapping the cycles. Another way of obtaining quiver gauge theories on a q-dimensional manifold M is to consider k coincident D$(q+r)$-branes wrapping the worldvolume manifold $X = M\times G/H$ where G/H is an r-dimensional homogeneous space for a Lie group G with a closed subgroup H. In the standard interpretation this system of D-branes corresponds to a rank k hermitean vector bundle $\mathcal{E}$ over X with a connection whose dynamics are governed by Yang-Mills gauge theory. For Kähler manifolds X the stability of such bundles (BPS conditions) is controlled by the Donaldson-Uhlenbeck-Yau (DUY) equations [11]. For G-equivariant bundles $\mathcal{E}\to X$ one finds that Yang-Mills theory on X reduces to a quiver gauge theory on M [12]–[15].

In this paper we will focus on some of these issues in quiver gauge theories on Kähler manifolds M which arise via a quotient by the natural action of the Lie group SU(2)×SU(2) on equivariant Chan-Paton bundles over $M\times\mathbb{C}P^1\times\mathbb{C}P^1$. Our analysis generalizes previous work on brane-antibrane systems from reduction on $M\times\mathbb{C}P^1$ [12, 16, 17], and on the generalization to chains of branes and antibranes arising from SU(2)-equivariant dimensional reduction on $M\times\mathbb{C}P^1$ [13, 18]. In particular, we will expand on the formalism introduced in [18] which merged the low-energy dynamics of brane-antibrane chains with quiver gauge theory into a Yang-Mills gauge theory of new objects on M termed "graded connections", which generalize the usual superconnections on the worldvolumes of coincident brane-antibrane pairs. This formalism is particularly well-suited to describe such physical instances and their novel effects, such as the equivalence between non-abelian quiver vortices on M and symmetric multi-instantons on the higher-dimensional space $M\times\mathbb{C}P^1\times\mathbb{C}P^1$. Moreover, when M is the noncommutative space $\mathbb{R}^{2n}_\theta$, it enables one to interpret noncommutative quiver solitons in the present case as states of D-branes in a straightforward manner, whilst providing a categorical approach to D-branes which characterizes their moduli beyond their K-theory charges. These quiver brane configurations require a more complex description than just that in terms of branes and antibranes, and we construct a category of D-branes which incorporates both their locations and their bindings to abelian magnetic monopoles.

The essential new ingredients of the present paper are that our quivers are of rank two, as opposed to the rank one quivers considered in [18], and the necessity of imposing relations on the quiver. The resulting quiver D-brane configuration is new, and comprises a two-dimensional lattice of branes and antibranes coupled to U(1)×U(1) Dirac monopole fields with interesting dynamics formulated through a higher-rank gauge theory of graded

JHEP09(2006)054

connections. We will also elaborate further on some of the constructions introduced in [18].

The outline of this paper is as follows. In section 2 we describe general features of the SU(2)×SU(2)-equivariant reduction of gauge theories on $M\times\mathbb{C}P^1\times\mathbb{C}P^1$ to an arbitrary Kähler manifold M, including the special case of the noncommutative euclidean space $M=\mathbb{R}^{2n}_\theta$. In section 3 we describe various features of the induced quiver gauge theory on M and develop the associated formalism of graded connections in this case. In section 4 we analyse the general structure of quiver gauge theory on M and the quiver vortex equations which describe the BPS sector. We then construct both BPS and non-BPS solutions of the Yang-Mills equations on the noncommutative space $\mathbb{R}^{2n}_\theta\times\mathbb{C}P^1\times\mathbb{C}P^1$, describe their induced quiver representations, and analyse in detail the structure of the moduli space of noncommutative instantons. Finally, in section 5 we realize our noncommutative instantons as configurations of D-branes by computing their topological charges, by computing their K-theory charges through a noncommutative equivariant version of the ABS construction, and by realizing them as objects in the category of quiver representations using some techniques of homological algebra.

2. Equivariant gauge theory

In this section we will analyse some aspects of SU(2)×SU(2)-equivariant gauge theory on spaces of the form $M\times\mathbb{C}P^1\times\mathbb{C}P^1$, where M is a Kähler manifold. After some preliminary definitions, we describe the equivariant decomposition of generic gauge bundles over $M\times\mathbb{C}P^1\times\mathbb{C}P^1$, and of their connections and curvatures. We then write down the corresponding Yang-Mills action functional and explain the generalization to noncommutative gauge theory. Equivariant dimensional reduction is described in general in [14], while general aspects of noncommutative field theories are reviewed in [19].

2.1 The Kähler manifold $M\times\mathbb{C}P^1\times\mathbb{C}P^1$

Let M be a Kähler manifold of real dimension $2n$ with local real coordinates $x=(x^\mu)\in\mathbb{R}^{2n}$, where the indices $\mu,\nu,\ldots$ run through $1,\ldots,2n$. Let $S^2_{(\ell)}\cong\mathbb{C}P^1_{(\ell)}$, $\ell=1,2$, be two copies of the standard two-sphere of constant radii R_ℓ with coordinates $\vartheta_\ell\in[0,\pi]$ and $\varphi_\ell\in[0,2\pi]$. We shall consider the product $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ which is also a Kähler manifold with local complex coordinates $(z^1,\ldots,z^n,y_1,y_2)\in\mathbb{C}^{n+2}$ and their complex conjugates, where

$$z^a \;=\; x^{2a-1}-\mathrm{i}\,x^{2a} \qquad\text{and}\qquad \bar z^{\bar a} \;=\; x^{2a-1}+\mathrm{i}\,x^{2a} \qquad\text{with}\quad a\;=\;1,\ldots,n \tag{2.1}$$

while

$$y_\ell \;=\; \frac{\sin\vartheta_\ell}{1+\cos\vartheta_\ell}\,\exp\left(-\mathrm{i}\,\varphi_\ell\right) \qquad\text{and}\qquad \bar y_\ell \;=\; \frac{\sin\vartheta_\ell}{1+\cos\vartheta_\ell}\,\exp\left(\mathrm{i}\,\varphi_\ell\right) \qquad\text{with}\quad \ell\;=\;1,2\ . \tag{2.2}$$

In these coordinates the riemannian metric

$$\mathrm{d}s^2=g_{\hat\mu\hat\nu}\,\mathrm{d}x^{\hat\mu}\,\mathrm{d}x^{\hat\nu} \tag{2.3}$$

on $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ takes the form

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + R_1^2\,\left(\mathrm{d}\vartheta_1^2+\sin^2\vartheta_1\,\mathrm{d}\varphi_1^2\right) + R_2^2\,\left(\mathrm{d}\vartheta_2^2+\sin^2\vartheta_2\,\mathrm{d}\varphi_2^2\right) \\ &= 2\,g_{a\bar b}\,\mathrm{d}z^a\,\mathrm{d}\bar z^{\bar b} + \frac{4\,R_1^2}{\left(1+y_1\bar y_1\right)^2}\,\mathrm{d}y_1\,\mathrm{d}\bar y_1 + \frac{4\,R_2^2}{\left(1+y_2\bar y_2\right)^2}\,\mathrm{d}y_2\,\mathrm{d}\bar y_2\,, \end{aligned} \tag{2.4}$$

where hatted indices $\hat\mu,\hat\nu,\ldots$ run over $1,\ldots,2n+4$. The Kähler two-form Ω is given by

$$\begin{aligned} \Omega &= \tfrac12\,\omega_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu + R_1^2\,\sin\vartheta_1\,\mathrm{d}\vartheta_1\wedge\mathrm{d}\varphi_1 + R_2^2\,\sin\vartheta_2\,\mathrm{d}\vartheta_2\wedge\mathrm{d}\varphi_2 \\ &= -2\,\mathrm{i}\,g_{a\bar b}\,\mathrm{d}z^a\wedge\mathrm{d}\bar z^{\bar b} - \frac{4\,\mathrm{i}\,R_1^2}{\left(1+y_1\bar y_1\right)^2}\,\mathrm{d}y_1\wedge\mathrm{d}\bar y_1 - \frac{4\,\mathrm{i}\,R_2^2}{\left(1+y_2\bar y_2\right)^2}\,\mathrm{d}y_2\wedge\mathrm{d}\bar y_2\ . \end{aligned} \tag{2.5}$$

2.2 Equivariant vector bundles

Let $\mathcal{E}\to M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ be a hermitean vector bundle of rank k. We wish to impose the condition of G-equivariance on this bundle with the group $G:=\mathrm{SU}(2)\times\mathrm{SU}(2)$ of rank 2 acting trivially on M and in the standard way on the homogeneous space $\mathbb{C}P^1\times\mathbb{C}P^1\cong G/H$, where $H:=\mathrm{U}(1)\times\mathrm{U}(1)$ is a maximal torus of G. This means that we should look for representations of the group G inside the $\mathrm{U}(k)$ structure group of the bundle $\mathcal{E}$, i.e. for k-dimensional unitary representations of G. For every pair of positive integers k_i and k_α, up to isomorphism there are unique irreducible $\mathrm{SU}(2)$-modules $\underline{V}_{k_i}$ and $\underline{V}_{k_\alpha}$ of dimensions k_i and k_α, respectively, and consequently a unique irreducible representation $\underline{V}_{k_{i\alpha}}:=\underline{V}_{k_i}\otimes\underline{V}_{k_\alpha}$ of G with dimension $k_{i\alpha}:=k_i\,k_\alpha$. Thus, for each pair of positive integers m_1 and m_2, the module

$$\underline{\mathcal{V}}\ =\ \bigoplus_{i=0}^{m_1}\,\bigoplus_{\alpha=0}^{m_2}\underline{V}_{k_{i\alpha}}\qquad\text{with}\quad\underline{V}_{k_{i\alpha}}\ \cong\ \mathbb{C}^{k_{i\alpha}}\qquad\text{and}\qquad\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}k_{i\alpha}\ =\ k \tag{2.6}$$

gives a representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ inside $\mathrm{U}(k)$. The structure group of the bundle $\mathcal{E}$ is correspondingly broken as

$$\mathrm{U}(k)\ \longrightarrow\ \prod_{i=0}^{m_1}\,\prod_{\alpha=0}^{m_2}\mathrm{U}(k_{i\alpha})\ . \tag{2.7}$$

As a result, we must construct bundles $\mathcal{E}\to M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ whose typical fibres $\underline{\mathcal{V}}$ are complex vector spaces with a direct sum decomposition as in (2.6). We will now describe how this is done explicitly.

There are natural equivalence functors between the categories of G-equivariant vector bundles over $M\times G/H$ and H-equivariant bundles over M, where H acts trivially on M [14]. If $E\to M$ is an H-equivariant bundle, then it defines a G-equivariant bundle $\mathcal{E}\to M\times\mathbb{C}P^1\times\mathbb{C}P^1$ by induction as

$$\mathcal{E}=G\times_H E\,, \tag{2.8}$$

where the H-action on $G\times E$ is given by $h\cdot(g,e)=(g\,h^{-1},h\cdot e)$ for $h\in H$, $g\in G$ and $e\in E$. We therefore focus our attention on the structure of H-equivariant bundles

$E \to M$. For this, it is more convenient to work in a holomorphic setting by passing to the universal complexification $G^{\rm c} := G \otimes \mathbb{C} = {\rm SL}(2,\mathbb{C})\times{\rm SL}(2,\mathbb{C})$ of the Lie group G. If $\mathcal{E} \to M\times\mathbb{C}P^1\times\mathbb{C}P^1$ is a G-equivariant vector bundle, then the G-action can be extended to an action of $G^{\rm c}$. Let $K = {\rm P}\times{\rm P}$ be the Borel subgroup of $G^{\rm c}$ with P the group of lower triangular matrices in ${\rm SL}(2,\mathbb{C})$. Its Levi decomposition is given by $K = U \ltimes H^{\rm c}$, where $H^{\rm c} := H \otimes \mathbb{C} = \mathbb{C}^\times\times\mathbb{C}^\times$. A representation $\underline{V}$ of K is irreducible if and only if the action of U on $\underline{V}$ is trivial and the restriction $\underline{V}|_{H^{\rm c}}$ is irreducible. It follows that there is a one-to-one correspondence between irreducible representations of K and irreducible representations of the Cartan subgroup $H^{\rm c} \subset G^{\rm c}$. The natural map $\mathbb{C}P^1\times\mathbb{C}P^1 = G/H \to G^{\rm c}/K$ is a diffeomorphism of projective varieties. The categorical equivalence above can then be reformulated as a one-to-one correspondence between $G^{\rm c}$-equivariant bundles $\mathcal{E} \to M\times\mathbb{C}P^1\times\mathbb{C}P^1$ and K-equivariant bundles over M, with K acting trivially on M.

The Lie algebra ${\rm sl}(2,\mathbb{C})$ is generated by the three Pauli matrices

$$\sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma_+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.9}$$

with the commutation relations

$$[\sigma_3\,,\,\sigma_\pm] \;=\; \pm\,2\,\sigma_\pm \qquad \text{and} \qquad [\sigma_+\,,\,\sigma_-] \;=\; \sigma_3\ . \tag{2.10}$$

The Lie algebra of U is generated by two independent copies of the element σ_-, while the Cartan subgroup $H^{\rm c}$ is generated by two independent copies of the element σ_3. For each $p \in \mathbb{Z}$ there is a unique irreducible representation $\underline{S}_p \cong \mathbb{C}$ of $\mathbb{C}^\times$ given by $\zeta\cdot v = \zeta^p\,v$ for $\zeta \in \mathbb{C}^\times$ and $v \in \underline{S}_p$. Thus for each pair of integers p_1, p_2 there is a unique irreducible module $\underline{S}^{(1)}_{p_1} \otimes \underline{S}^{(2)}_{p_2} \cong \mathbb{C}$ over the subgroup $H^{\rm c} = \mathbb{C}^\times_{(1)}\times\mathbb{C}^\times_{(2)}$. Since the manifold M carries the trivial action of the group $H^{\rm c}$, any K-equivariant bundle $E \to M$ admits a finite Whitney sum decomposition into isotopical components as $E = \bigoplus_{p_1,p_2} E_{p_1\,p_2} \otimes \underline{S}^{(1)}_{p_1} \otimes \underline{S}^{(2)}_{p_2}$, where the sum runs over the set of eigenvalues for the $H^{\rm c}$-action on E and $E_{p_1\,p_2} \to M$ are bundles with the trivial $H^{\rm c}$-action. From the commutation relations (2.10) it follows that the U-action on $E_{p_1\,p_2} \otimes \underline{S}^{(1)}_{p_1} \otimes \underline{S}^{(2)}_{p_2}$ corresponds to independent bundle morphisms $E_{p_1\,p_2} \to E_{p_1-2\,p_2}$ and $E_{p_1\,p_2} \to E_{p_1\,p_2-2}$, along with the trivial σ_--actions on the irreducible $H^{\rm c}$-modules $\underline{S}^{(1)}_{p_1} \otimes \underline{S}^{(2)}_{p_2}$.

After an appropriate twist by an $H^{\rm c}$-module and a relabelling, the σ_3-actions are given by the $H^{\rm c}$-equivariant decomposition

$$E = \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} E_{k_{i\alpha}} \otimes \underline{S}^{(1)}_{m_1-2i} \otimes \underline{S}^{(2)}_{m_2-2\alpha}\,, \tag{2.11}$$

while the U-action is determined through the diagram

$$\begin{array}{ccccccc}
E_{k_{m_1\,0}} & \xrightarrow{\phi^{(1)}_{m_1\,0}} & E_{k_{m_1-1\,0}} & \xrightarrow{\phi^{(1)}_{m_1-1\,0}} & \cdots & \xrightarrow{\phi^{(1)}_{10}} & E_{k_{00}} \\
\phi^{(2)}_{m_1\,1}\uparrow & & \phi^{(2)}_{m_1-1\,1}\uparrow & & & & \uparrow\phi^{(2)}_{01} \\
\vdots & & \vdots & & & & \vdots \\
\phi^{(2)}_{m_1\,m_2-1}\uparrow & & \phi^{(2)}_{m_1-1\,m_2-1}\uparrow & & & & \uparrow\phi^{(2)}_{0\,m_2-1} \\
E_{k_{m_1\,m_2-1}} & \xrightarrow{\phi^{(1)}_{m_1\,m_2-1}} & E_{k_{m_1-1\,m_2-1}} & \xrightarrow{\phi^{(1)}_{m_1-1\,m_2-1}} & \cdots & \xrightarrow{\phi^{(1)}_{1\,m_2-1}} & E_{k_{0\,m_2-1}} \\
\phi^{(2)}_{m_1 m_2}\uparrow & & \phi^{(2)}_{m_1-1\,m_2}\uparrow & & & & \uparrow\phi^{(2)}_{0m_2} \\
E_{k_{m_1\,m_2}} & \xrightarrow[\phi^{(1)}_{m_1 m_2}]{} & E_{k_{m_1-1\,m_2}} & \xrightarrow[\phi^{(1)}_{m_1-1\,m_2}]{} & \cdots & \xrightarrow[\phi^{(1)}_{1\,m_2}]{} & E_{k_{0\,m_2}}
\end{array} \tag{2.12}$$

of holomorphic bundle maps with $\phi^{(1)}_{m_1+1\,\alpha} = 0 = \phi^{(1)}_{0\alpha}$ for $\alpha = 0, 1, \ldots, m_2$ and $\phi^{(2)}_{i\,m_2+1} = 0 = \phi^{(2)}_{i0}$ for $i = 0, 1, \ldots, m_1$. Since the Lie algebra of U is abelian, these maps generate a *commutative* bundle diagram (2.12), i.e. for each i, α one has

$$\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} = \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1}\ . \tag{2.13}$$

Finally, we can now consider the underlying H-equivariant hermitean vector bundle and introduce the standard p_ℓ-monopole line bundles

$$\mathcal{L}^{p_\ell}_{(\ell)} = \mathrm{SU}(2)\times_{\mathrm{U}(1)} \underline{S}^{(\ell)}_{p_\ell} \tag{2.14}$$

over the homogeneous spaces $\mathbb{C}P^1_{(\ell)}$ for $\ell = 1, 2$. Then the original rank k hermitean vector bundle (2.8) over $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ admits an equivariant decomposition

$$\mathcal{E} \;=\; \bigoplus_{i=0}^{m_1}\,\bigoplus_{\alpha=0}^{m_2}\,\mathcal{E}_{i\alpha} \qquad \text{with} \quad \mathcal{E}_{i\alpha} \;=\; E_{k_{i\alpha}} \otimes \mathcal{L}^{m_1-2i}_{(1)} \otimes \mathcal{L}^{m_2-2\alpha}_{(2)}\,, \tag{2.15}$$

where $E_{k_{i\alpha}} \to M$ is a hermitean vector bundle of rank $k_{i\alpha}$ with typical fibre the module $\underline{V}_{k_{i\alpha}}$ in (2.6), and $\mathcal{E}_{i\alpha} \to M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ is the bundle with fibres

$$\big(\mathcal{E}_{i\alpha}\big)_{(x,\,y_1,\,\bar y_1,\,y_2,\,\bar y_2)} = \big(E_{k_{i\alpha}}\big)_x \otimes \big(\mathcal{L}^{m_1-2i}_{(1)}\big)_{(y_1,\bar y_1)} \otimes \big(\mathcal{L}^{m_2-2\alpha}_{(2)}\big)_{(y_2,\,\bar y_2)}\ . \tag{2.16}$$

2.3 Equivariant gauge fields

Let $\mathcal{A}$ be a connection on the hermitean vector bundle $\mathcal{E} \to M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ having the form $\mathcal{A} = \mathcal{A}_{\hat\mu}\,\mathrm{d}x^{\hat\mu}$ in local coordinates $(x^{\hat\mu})$ and taking values in the Lie algebra $\mathrm{u}(k)$. We will now describe the G-equivariant reduction of $\mathcal{A}$ on $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$. The spherical dependences are completely determined by the unique SU(2)-invariant connections $a^{(\ell)}_{p_\ell}$, $\ell = 1, 2$, on the monopole line bundles (2.14) having, in local complex coordinates on $\mathbb{C}P^1_{(\ell)}$, the forms

$$a^{(\ell)}_{p_\ell} = \frac{p_\ell}{2\,(1+y_\ell\bar y_\ell)}\ (\bar y_\ell\,\mathrm{d}y_\ell - y_\ell\,\mathrm{d}\bar y_\ell)\ . \tag{2.17}$$

The curvatures of these connections are

$$f^{(\ell)}_{p_\ell} \;=\; \mathrm{d}a^{(\ell)}_{p_\ell} \;=\; -\frac{p_\ell}{(1+y_\ell\bar{y}_\ell)^2}\,\mathrm{d}y_\ell\wedge\mathrm{d}\bar{y}_\ell\,, \tag{2.18}$$

and their topological charges are given by the degrees of the complex line bundles $\mathcal{L}^{p_\ell}_{(\ell)}\to\mathbb{C}P^1_{(\ell)}$ as

$$\deg\,\mathcal{L}^{p_\ell}_{(\ell)} \;=\; \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}P^1_{(\ell)}} f^{(\ell)}_{p_\ell} \;=\; p_\ell\ . \tag{2.19}$$

In the spherical coordinates $(\vartheta_\ell,\varphi_\ell)\in S^2_{(\ell)}$ the monopole fields can be written as

$$a^{(\ell)}_{p_\ell} \;=\; -\frac{\mathrm{i}\,p_\ell}{2}\,(1-\cos\vartheta_\ell)\,\mathrm{d}\varphi_\ell \qquad\text{and}\qquad f^{(\ell)}_{p_\ell} \;=\; \mathrm{d}a^{(\ell)}_{p_\ell} \;=\; -\frac{\mathrm{i}\,p_\ell}{2}\,\sin\vartheta_\ell\,\mathrm{d}\vartheta_\ell\wedge\mathrm{d}\varphi_\ell\ . \tag{2.20}$$

Related to the monopole fields are the unique, covariantly constant SU(2)-invariant forms of types $(1,0)$ and $(0,1)$ on $\mathbb{C}P^1_{(\ell)}$ given respectively by

$$\beta_\ell \;=\; \frac{\mathrm{d}y_\ell}{1+y_\ell\bar{y}_\ell} \qquad\text{and}\qquad \bar{\beta}_\ell \;=\; \frac{\mathrm{d}\bar{y}_\ell}{1+y_\ell\bar{y}_\ell}\ . \tag{2.21}$$

They take values respectively in the components $\mathcal{L}^2_{(\ell)}$ and $\mathcal{L}^{-2}_{(\ell)}$ of the complexified cotangent bundle $T^*\mathbb{C}P^1_{(\ell)}\otimes\mathbb{C}=\mathcal{L}^2_{(\ell)}\oplus\mathcal{L}^{-2}_{(\ell)}$ over $\mathbb{C}P^1_{(\ell)}$. Note that there is no summation over the index ℓ in (2.17)–(2.21).

With respect to the isotopical decomposition (2.15), the twisted $\mathrm{u}(k)$-valued gauge potential $\mathcal{A}$ thus splits into $k_{i\alpha}\times k_{j\beta}$ blocks as

$$\mathcal{A} \;=\; \Big(\mathcal{A}^{i\alpha,\,j\beta}\Big) \qquad\text{with}\quad \mathcal{A}^{i\alpha,\,j\beta}\;\in\;\mathrm{Hom}\big(\underline{V}_{k_{j\beta}}\,,\underline{V}_{k_{i\alpha}}\big)\,, \tag{2.22}$$

where

$$\mathcal{A}^{i\alpha,\,i\alpha} \;=\; A^{i\alpha}(x)\otimes 1\otimes 1+\mathbf{1}_{k_{i\alpha}}\otimes\Big(a^{(1)}_{m_1-2i}(y_1)\otimes 1+1\otimes a^{(2)}_{m_2-2\alpha}(y_2)\Big)\ , \tag{2.23}$$

$$\mathcal{A}^{i\alpha,\,i+1\,\alpha} =: \quad \Phi^{(1)}_{i+1\,\alpha} \quad = \quad \phi^{(1)}_{i+1\,\alpha}(x)\otimes\bar{\beta}_1(y_1)\otimes 1\ , \tag{2.24}$$

$$\mathcal{A}^{i+1\,\alpha,\,i\alpha} \;=\; -\big(\mathcal{A}^{i\alpha,\,i+1\,\alpha}\big)^\dagger \;=\; -\big(\phi^{(1)}_{i+1\,\alpha}(x)\big)^\dagger\otimes\beta_1(y_1)\otimes 1\ , \tag{2.25}$$

$$\mathcal{A}^{i\alpha,\,i\,\alpha+1} =: \quad \Phi^{(2)}_{i\,\alpha+1} \quad = \quad \phi^{(2)}_{i\,\alpha+1}(x)\otimes 1\otimes\bar{\beta}_2(y_2)\ , \tag{2.26}$$

$$\mathcal{A}^{i\,\alpha+1,\,i\alpha} \;=\; -\big(\mathcal{A}^{i\alpha,\,i\,\alpha+1}\big)^\dagger \;=\; -\big(\phi^{(2)}_{i\,\alpha+1}(x)\big)^\dagger\otimes 1\otimes\beta_2(y_2)\ . \tag{2.27}$$

All other components $\mathcal{A}^{i\alpha,\,j\beta}$ vanish, while the bundle morphisms $\Phi^{(1)}_{i+1\,\alpha}\in\mathrm{Hom}(\mathcal{E}_{i+1\,\alpha},\mathcal{E}_{i\alpha})$ and $\Phi^{(2)}_{i\,\alpha+1}\in\mathrm{Hom}(\mathcal{E}_{i\,\alpha+1},\mathcal{E}_{i\alpha})$ obey $\Phi^{(1)}_{m_1+1\,\alpha}=0=\Phi^{(1)}_{0\alpha}$ for $\alpha=0,1,\dots,m_2$ and $\Phi^{(2)}_{i\,m_2+1}=0=\Phi^{(2)}_{i0}$ for $i=0,1,\dots,m_1$. The gauge potentials $A^{i\alpha}\in\mathrm{u}(k_{i\alpha})$ are connections on the hermitean vector bundles $E_{k_{i\alpha}}\to M$, while the bi-fundamental scalar fields $\phi^{(1)}_{i+1\,\alpha}$ and $\phi^{(2)}_{i\,\alpha+1}$ transform in the representations $\underline{V}_{k_{i\alpha}}\otimes\underline{V}^\vee_{k_{i+1\,\alpha}}$ and $\underline{V}_{k_{i\alpha}}\otimes\underline{V}^\vee_{k_{i\,\alpha+1}}$ of the subgroups $\mathrm{U}(k_{i\alpha})\times\mathrm{U}(k_{i+1\,\alpha})$ and $\mathrm{U}(k_{i\alpha})\times\mathrm{U}(k_{i\,\alpha+1})$ of the original $\mathrm{U}(k)$ gauge group.

The curvature two-form $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ of the connection $\mathcal{A}$ has components $\mathcal{F}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\mathcal{A}_{\hat{\nu}} - \partial_{\hat{\nu}}\mathcal{A}_{\hat{\mu}} + [\mathcal{A}_{\hat{\mu}}, \mathcal{A}_{\hat{\nu}}]$ in local coordinates $(x^{\hat{\mu}})$, where $\partial_{\hat{\mu}} := \partial/\partial x^{\hat{\mu}}$. It also take values in the Lie algebra $\mathrm{u}(k)$, and in local coordinates on $M \times \mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$ it can be written as

$$\begin{aligned}\mathcal{F} = {} & \tfrac{1}{2}\,\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \mathcal{F}_{\mu y_1}\,\mathrm{d}x^\mu \wedge \mathrm{d}y_1 + \mathcal{F}_{\mu \bar{y}_1}\,\mathrm{d}x^\mu \wedge \mathrm{d}\bar{y}_1 + \mathcal{F}_{\mu y_2}\,\mathrm{d}x^\mu \wedge \mathrm{d}y_2 \\ & + \mathcal{F}_{\mu \bar{y}_2}\,\mathrm{d}x^\mu \wedge \mathrm{d}\bar{y}_2 + \mathcal{F}_{y_1 \bar{y}_1}\,\mathrm{d}y_1 \wedge \mathrm{d}\bar{y}_1 + \mathcal{F}_{y_2 \bar{y}_2}\,\mathrm{d}y_2 \wedge \mathrm{d}\bar{y}_2 + \mathcal{F}_{y_1 y_2}\,\mathrm{d}y_1 \wedge \mathrm{d}y_2 \\ & + \mathcal{F}_{\bar{y}_1 \bar{y}_2}\,\mathrm{d}\bar{y}_1 \wedge \mathrm{d}\bar{y}_2 + \mathcal{F}_{y_1 \bar{y}_2}\,\mathrm{d}y_1 \wedge \mathrm{d}\bar{y}_2 + \mathcal{F}_{\bar{y}_1 y_2}\,\mathrm{d}\bar{y}_1 \wedge \mathrm{d}y_2\ . \end{aligned} \tag{2.28}$$

The calculation of the curvature (2.28) for $\mathcal{A}$ of the form (2.22)–(2.27) yields

$$\mathcal{F} \;=\; \Big(\mathcal{F}^{i\alpha,\,j\beta}\Big) \qquad \text{with} \quad \mathcal{F}^{i\alpha,\,j\beta} \;=\; \mathrm{d}\mathcal{A}^{i\alpha,\,j\beta} + \sum_{l=0}^{m_1}\,\sum_{\gamma=0}^{m_2} \mathcal{A}^{i\alpha,\,l\gamma} \wedge \mathcal{A}^{l\gamma,\,j\beta}\,, \tag{2.29}$$

where

$$\begin{aligned}\mathcal{F}^{i\alpha,\,i\alpha} = {} & F^{i\alpha} + f^{(1)}_{m_1-2i} + f^{(2)}_{m_2-2\alpha} \\ & + \Big(\phi^{(1)}_{i+1\,\alpha}\,\big(\phi^{(1)}_{i+1\,\alpha}\big)^\dagger - \big(\phi^{(1)}_{i\alpha}\big)^\dagger\,\phi^{(1)}_{i\alpha}\Big)\,\big(\beta_1 \wedge \bar{\beta}_1\big) \\ & + \Big(\phi^{(2)}_{i\,\alpha+1}\,\big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger - \big(\phi^{(2)}_{i\alpha}\big)^\dagger\,\phi^{(2)}_{i\alpha}\Big)\,\big(\beta_2 \wedge \bar{\beta}_2\big)\ , \end{aligned} \tag{2.30}$$

$$\mathcal{F}^{i\alpha,\,i+1\,\alpha} = D\phi^{(1)}_{i+1\,\alpha} \wedge \bar{\beta}_1\,, \tag{2.31}$$

$$\mathcal{F}^{i+1\,\alpha,\,i\alpha} = -\big(\mathcal{F}^{i\alpha,\,i+1\,\alpha}\big)^\dagger \;=\; -\big(D\phi^{(1)}_{i+1\,\alpha}\big)^\dagger \wedge \beta_1\,, \tag{2.32}$$

$$\mathcal{F}^{i\alpha,\,i\,\alpha+1} = D\phi^{(2)}_{i\,\alpha+1} \wedge \bar{\beta}_2\,, \tag{2.33}$$

$$\mathcal{F}^{i\,\alpha+1,\,i\alpha} = -\big(\mathcal{F}^{i\alpha,\,i\,\alpha+1}\big)^\dagger \;=\; -\big(D\phi^{(2)}_{i\,\alpha+1}\big)^\dagger \wedge \beta_2\,, \tag{2.34}$$

$$\mathcal{F}^{i\alpha,\,i+1\,\alpha+1} = \Big(\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1}\Big)\,\bar{\beta}_1 \wedge \bar{\beta}_2\,, \tag{2.35}$$

$$\mathcal{F}^{i+1\,\alpha+1,\,i\,\alpha} = -\big(\mathcal{F}^{i\alpha,\,i+1\,\alpha+1}\big)^\dagger \;=\; -\Big(\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1}\Big)^\dagger\,\beta_1 \wedge \beta_2\,, \tag{2.36}$$

$$\mathcal{F}^{i\,\alpha+1,\,i+1\,\alpha} = \Big(\big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger\,\phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha+1}\,\big(\phi^{(2)}_{i+1\,\alpha+1}\big)^\dagger\Big)\,\bar{\beta}_1 \wedge \beta_2\,, \tag{2.37}$$

$$\mathcal{F}^{i+1\,\alpha,\,i\,\alpha+1} = -\big(\mathcal{F}^{i\,\alpha+1,\,i+1\,\alpha}\big)^\dagger \;=\; \Big(\big(\phi^{(1)}_{i+1\alpha}\big)^\dagger\,\phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i+1\,\alpha+1}\big(\phi^{(1)}_{i+1\alpha+1}\big)^\dagger\Big)\,\bar{\beta}_2 \wedge \beta_1 \tag{2.38}$$

with all other components vanishing. We have suppressed the tensor product structure pertaining to $M \times \mathbb{C}P^1 \times \mathbb{C}P^1$ in (2.30)–(2.38). Here $F^{i\alpha} := \mathrm{d}A^{i\alpha} + A^{i\alpha} \wedge A^{i\alpha} = \frac{1}{2}\,F^{i\alpha}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ are the curvatures of the bundles $E_{k_{i\alpha}} \to M$, and we have introduced the bi-fundamental covariant derivatives

$$D\phi^{(1)}_{i+1\,\alpha} := \mathrm{d}\phi^{(1)}_{i+1\,\alpha} + A^{i\alpha}\,\phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha}\,A^{i+1\,\alpha}\,, \tag{2.39}$$

$$D\phi^{(2)}_{i\,\alpha+1} := \mathrm{d}\phi^{(2)}_{i\,\alpha+1} + A^{i\alpha}\,\phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,A^{i\,\alpha+1}\ . \tag{2.40}$$

From (2.30)–(2.38) we find the non-vanishing field strength components

$$\mathcal{F}_{\mu\nu}^{i\alpha,\,i\alpha} = F_{\mu\nu}^{i\alpha}\,, \tag{2.41}$$

$$\mathcal{F}_{\mu\bar{y}_1}^{i\alpha,\,i+1\,\alpha} = \frac{1}{1+y_1\bar{y}_1}\,D_\mu\phi_{i+1\,\alpha}^{(1)} \;=\; -\left(\mathcal{F}_{\mu y_1}^{i+1\,\alpha,\,i\alpha}\right)^\dagger, \tag{2.42}$$

$$\mathcal{F}_{\mu\bar{y}_2}^{i\alpha,\,i\,\alpha+1} = \frac{1}{1+y_2\bar{y}_2}\,D_\mu\phi_{i\,\alpha+1}^{(2)} \;=\; -\left(\mathcal{F}_{\mu y_2}^{i\,\alpha+1,\,i\alpha}\right)^\dagger, \tag{2.43}$$

$$\mathcal{F}_{y_1\bar{y}_1}^{i\alpha,\,i\alpha} = -\frac{1}{(1+y_1\bar{y}_1)^2}\,\Big((m_1-2i)\;\mathbf{1}_{k_{i\alpha}} + \left(\phi_{i\alpha}^{(1)}\right)^\dagger\phi_{i\alpha}^{(1)} - \phi_{i+1\,\alpha}^{(1)}\left(\phi_{i+1\,\alpha}^{(1)}\right)^\dagger\Big)\,, \tag{2.44}$$

$$\mathcal{F}_{y_2\bar{y}_2}^{i\alpha,\,i\alpha} = -\frac{1}{(1+y_2\bar{y}_2)^2}\,\Big((m_2-2\alpha)\;\mathbf{1}_{k_{i\alpha}} + \left(\phi_{i\alpha}^{(2)}\right)^\dagger\phi_{i\alpha}^{(2)} - \phi_{i\,\alpha+1}^{(2)}\left(\phi_{i\,\alpha+1}^{(2)}\right)^\dagger\Big) \tag{2.45}$$

and

$$\mathcal{F}_{\bar{y}_1\bar{y}_2}^{i\alpha,\,i+1\,\alpha+1} = \frac{\phi_{i+1\,\alpha}^{(1)}\,\phi_{i+1\,\alpha+1}^{(2)} - \phi_{i\,\alpha+1}^{(2)}\,\phi_{i+1\,\alpha+1}^{(1)}}{(1+y_1\bar{y}_1)\,(1+y_2\bar{y}_2)} \;=\; -\left(\mathcal{F}_{y_1y_2}^{i+1\,\alpha+1,\,i\alpha}\right)^\dagger, \tag{2.46}$$

$$\mathcal{F}_{y_1\bar{y}_2}^{i\,\alpha+1,\,i+1\,\alpha} = \frac{\left(\phi_{i\,\alpha+1}^{(2)}\right)^\dagger\phi_{i+1\,\alpha}^{(1)} - \phi_{i+1\,\alpha+1}^{(1)}\left(\phi_{i+1\,\alpha+1}^{(2)}\right)^\dagger}{(1+y_1\bar{y}_1)\,(1+y_2\bar{y}_2)} \;=\; -\left(\mathcal{F}_{\bar{y}_1y_2}^{i+1\,\alpha,\,i\,\alpha+1}\right)^\dagger. \tag{2.47}$$

Note that at this stage we do not generally require the imposition of the holomorphic constraints (2.13) in this ansatz, which ensure that the bundle diagram (2.12) commutes. Later on we will see that they arise as a *dynamical* constraint for BPS solutions of the Yang-Mills equations on $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ that force the vanishing of the cross-components (2.46) of the field strength tensor between the two copies of the sphere. In fact, our particular ansatz in the noncommutative gauge theory will automatically satisfy this condition, as well as the analogous ones which force the cross-components (2.47) to vanish.

2.4 The Yang-Mills functional

Let us now consider the equivariant reduction of the Yang-Mills lagrangian

$$\begin{aligned} L_{\rm YM} &:= -\tfrac{1}{4}\,\sqrt{\hat{g}}\;\mathrm{tr}_{k\times k}\;\mathcal{F}_{\hat{\mu}\hat{\nu}}\,\mathcal{F}^{\hat{\mu}\hat{\nu}} \\ &= -\tfrac{1}{4}\,\sqrt{\hat{g}}\;\mathrm{tr}_{k\times k}\Big\{\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} + g^{\mu\nu}\,g^{y_1\bar{y}_1}\;(\mathcal{F}_{\mu y_1}\,\mathcal{F}_{\nu\bar{y}_1} + \mathcal{F}_{\mu\bar{y}_1}\,\mathcal{F}_{\nu y_1}) \\ &\quad + g^{\mu\nu}\,g^{y_2\bar{y}_2}\;(\mathcal{F}_{\mu y_2}\,\mathcal{F}_{\nu\bar{y}_2} + \mathcal{F}_{\mu\bar{y}_2}\,\mathcal{F}_{\nu y_2}) - 2\,\left(g^{y_1\bar{y}_1}\,\mathcal{F}_{y_1\bar{y}_1}\right)^2 - 2\,\left(g^{y_2\bar{y}_2}\,\mathcal{F}_{y_2\bar{y}_2}\right)^2 \\ &\quad + 2\,g^{y_1\bar{y}_1}\,g^{y_2\bar{y}_2}\;(\mathcal{F}_{\bar{y}_1\bar{y}_2}\,\mathcal{F}_{y_1y_2} + \mathcal{F}_{y_1y_2}\,\mathcal{F}_{\bar{y}_1\bar{y}_2} + \mathcal{F}_{\bar{y}_1y_2}\,\mathcal{F}_{y_1\bar{y}_2} + \mathcal{F}_{y_1\bar{y}_2}\,\mathcal{F}_{\bar{y}_1y_2})\Big\}, \end{aligned} \tag{2.48}$$

where $\hat{g} = \det(g_{\hat{\mu}\hat{\nu}}) = g\;g_{\mathbb{C}P^1_{(1)}}\;g_{\mathbb{C}P^1_{(2)}}$ with $g = \det(g_{\mu\nu})$ and

$$\sqrt{g_{\mathbb{C}P^1_{(\ell)}}} \;=\; \frac{2\,R_\ell^2}{(1+y_\ell\bar{y}_\ell)^2} \;=\; \left(g^{y_\ell\bar{y}_\ell}\right)^{-1}\,. \tag{2.49}$$

For the ansatz of the section 2.3 above we substitute (2.41)–(2.47). After integration over the spherical factors $\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$, the dimensional reduction of the corresponding Yang-

Mills action functional is given by

$$
\begin{aligned}
S_{\rm YM} &:= \int_{M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}} \mathrm{d}^{2n+4}x\ L_{\rm YM} \\
&= \pi\, R_1^2\, R_2^2 \int_M \mathrm{d}^{2n}x\ \sqrt{g}\ \sum_{i=0}^{m_1}\ \sum_{\alpha=0}^{m_2} \mathrm{tr}_{k_{i\alpha}\times k_{i\alpha}} \Big[\big(F^{i\alpha}_{\mu\nu}\big)^\dagger\, F^{i\alpha\,\mu\nu} \\
&\quad + \frac{1}{R_1^2}\, \big(D_\mu \phi^{(1)}_{i+1\,\alpha}\big)\, \big(D^\mu \phi^{(1)}_{i+1\,\alpha}\big)^\dagger + \frac{1}{R_1^2}\, \big(D_\mu \phi^{(1)}_{i\alpha}\big)^\dagger\, \big(D^\mu \phi^{(1)}_{i\alpha}\big) \\
&\quad + \frac{1}{R_2^2}\, \big(D_\mu \phi^{(2)}_{i\,\alpha+1}\big)\, \big(D^\mu \phi^{(2)}_{i\,\alpha+1}\big)^\dagger + \frac{1}{R_2^2}\, \big(D_\mu \phi^{(2)}_{i\alpha}\big)^\dagger\, \big(D^\mu \phi^{(2)}_{i\alpha}\big) \\
&\quad + \frac{1}{2\,R_1^4}\, \Big((m_1 - 2i)\ \mathbf{1}_{k_{i\alpha}} + \big(\phi^{(1)}_{i\alpha}\big)^\dagger\, \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i+1\,\alpha}\, \big(\phi^{(1)}_{i+1\,\alpha}\big)^\dagger \Big)^2 \\
&\quad + \frac{1}{2\,R_2^4}\, \Big((m_2 - 2\alpha)\ \mathbf{1}_{k_{i\alpha}} + \big(\phi^{(2)}_{i\alpha}\big)^\dagger\, \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i\,\alpha+1}\, \big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger \Big)^2 \\
&\quad + \frac{1}{2\,R_1^2\,R_2^2}\, \Big(\phi^{(1)}_{i+1\,\alpha}\, \phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\, \phi^{(1)}_{i+1\,\alpha+1} \Big)\, \Big(\phi^{(1)}_{i+1\,\alpha}\, \phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\, \phi^{(1)}_{i+1\,\alpha+1} \Big)^\dagger \\
&\quad + \frac{1}{2\,R_1^2\,R_2^2}\, \Big(\phi^{(1)}_{i\,\alpha-1}\, \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i-1\,\alpha}\, \phi^{(1)}_{i\alpha} \Big)^\dagger\, \Big(\phi^{(1)}_{i\,\alpha-1}\, \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i-1\,\alpha}\, \phi^{(1)}_{i\alpha} \Big) \\
&\quad + \frac{1}{2\,R_1^2\,R_2^2}\, \Big(\big(\phi^{(2)}_{i\alpha}\big)^\dagger\, \phi^{(1)}_{i+1\,\alpha-1} - \phi^{(1)}_{i+1\,\alpha}\, \big(\phi^{(2)}_{i+1\,\alpha}\big)^\dagger \Big)\, \Big(\big(\phi^{(2)}_{i\alpha}\big)^\dagger\, \phi^{(1)}_{i+1\,\alpha-1} - \phi^{(1)}_{i+1\,\alpha}\, \big(\phi^{(2)}_{i+1\,\alpha}\big)^\dagger \Big)^\dagger \\
&\quad + \frac{1}{2\,R_1^2\,R_2^2}\, \Big(\big(\phi^{(2)}_{i-1\,\alpha+1}\big)^\dagger\, \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i\,\alpha+1}\, \big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger \Big)^\dagger\, \Big(\big(\phi^{(2)}_{i-1\,\alpha+1}\big)^\dagger\, \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i\,\alpha+1}\, \big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger \Big) \Big]\ .
\end{aligned}
\tag{2.50}
$$

All individual terms in (2.50) are $k_{i\alpha}\times k_{i\alpha}$ matrices. Recall that $\phi^{(1)}_{i+1\,\alpha}$ are $k_{i\alpha}\times k_{i+1\,\alpha}$ matrices, $\phi^{(2)}_{i\,\alpha+1}$ are $k_{i\alpha}\times k_{i\,\alpha+1}$ matrices and $A^{i\alpha}_\mu$ are $k_{i\alpha}\times k_{i\alpha}$ matrices. The action (2.50) is non-negative, and it can be regarded as an energy functional for static fields on $\mathbb{R}^{0,1}\times M$ in the temporal gauge.

2.5 Noncommutative gauge theory

When we come to construct explicit solutions of the Yang-Mills equations we will specialize to the Kähler manifold $M = \mathbb{R}^{2n}$ with metric tensor $g_{\mu\nu} = \delta_{\mu\nu}$ and pass to a noncommutative deformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}_\theta$. The spherical factors $\mathbb{C}P^1_{(\ell)}$, $\ell = 1, 2$, will always remain commutative spaces. The noncommutative space $\mathbb{R}^{2n}_\theta$ is defined by declaring its coordinate functions $\hat{x}^1, \dots, \hat{x}^{2n}$ to obey the Heisenberg algebra relations

$$
[\,\hat{x}^\mu\,,\,\hat{x}^\nu\,] = \mathrm{i}\,\theta^{\mu\nu}
\tag{2.51}
$$

with a constant real antisymmetric tensor $\theta^{\mu\nu}$ of maximal rank n. Via an orthogonal transformation of the coordinates, the matrix $\theta = (\theta^{\mu\nu})$ can be rotated into its canonical block-diagonal form with non-vanishing components

$$
\theta^{2a-1\ 2a} \ =\ -\theta^{2a\ 2a-1} \ =:\ \theta^a
\tag{2.52}
$$

for $a = 1, \dots, n$. We will assume for definiteness that all $\theta^a > 0$. The noncommutative version of the complex coordinates (2.1) has the non-vanishing commutators

$$
\big[\,\hat{z}^a\,,\,\hat{\bar{z}}^{\bar{b}}\,\big] \ =\ -2\,\delta^{a\bar{b}}\,\theta^a \ =:\ \theta^{a\bar{b}} \ =\ -\theta^{\bar{b}a} \ <\ 0\ .
\tag{2.53}
$$

Taking the product of $\mathbb{R}^{2n}_\theta$ with the commutative spheres $\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ means extending the noncommutativity matrix θ by vanishing entries along the four new directions.

The algebra (2.51) can be represented on the Fock space $\mathcal{H}$ which may be realized as the linear span

$$\mathcal{H} = \bigoplus_{r_1,\dots,r_n=0}^{\infty} \mathbb{C}|r_1,\dots,r_n\rangle\,, \tag{2.54}$$

where the orthonormal basis states

$$|r_1,\dots,r_n\rangle = \prod_{a=1}^{n} (2\,\theta^a\,r_a!)^{-1/2}\ (\hat{z}^a)^{r_a}|0,\dots,0\rangle \tag{2.55}$$

are connected by the action of creation and annihilation operators subject to the commutation relations

$$\Big[\,\frac{\hat{\bar{z}}^{\bar{b}}}{\sqrt{2\,\theta^b}}\,,\ \frac{\hat{z}^a}{\sqrt{2\,\theta^a}}\,\Big] = \delta^{a\bar{b}}\ . \tag{2.56}$$

In the Weyl operator realization $f\mapsto\hat{f}$ which maps Schwartz functions f on $\mathbb{R}^{2n}$ into compact operators $\hat{f}$ on $\mathcal{H}$, coordinate derivatives are given by inner derivations of the noncommutative algebra according to

$$\widehat{\partial_{z^a}f}\ =\ \theta_{a\bar{b}}\,\big[\hat{\bar{z}}^{\bar{b}}\,,\,\hat{f}\,\big]\ =:\ \partial_{\hat{z}^a}\hat{f} \qquad\text{and}\qquad \widehat{\partial_{\bar{z}^{\bar{a}}}f}\ =\ \theta_{\bar{a}b}\,\big[\hat{z}^b\,,\,\hat{f}\,\big]\ =:\ \partial_{\hat{\bar{z}}^{\bar{a}}}\hat{f}\,, \tag{2.57}$$

where $\theta_{a\bar{b}}$ is defined via $\theta_{b\bar{c}}\,\theta^{\bar{c}a}=\delta^a_b$ so that $\theta_{a\bar{b}}=-\theta_{\bar{b}a}=\frac{\delta_{a\bar{b}}}{2\,\theta^a}$. On the other hand, integrals are given by traces over the Fock space $\mathcal{H}$ as

$$\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}x\ f(x) = \mathrm{Pf}(2\pi\,\theta)\ \mathrm{Tr}_{\mathcal{H}}\ \hat{f}\ . \tag{2.58}$$

Vector bundles $E\to\mathbb{R}^{2n}$ whose typical fibres are complex vector spaces $\underline{V}$ are replaced by the corresponding (trivial) projective modules $\underline{V}\otimes\mathcal{H}$ over $\mathbb{R}^{2n}_\theta$. The field strength components along $\mathbb{R}^{2n}_\theta$ in (2.28) read $\hat{\mathcal{F}}_{\mu\nu} = \partial_{\hat{x}^\mu}\hat{\mathcal{A}}_\nu - \partial_{\hat{x}^\nu}\hat{\mathcal{A}}_\mu + [\hat{\mathcal{A}}_\mu,\hat{\mathcal{A}}_\nu]$, where $\hat{\mathcal{A}}_\mu$ are simultaneously valued in $\mathrm{u}(k)$ and in $\mathrm{End}(\mathcal{H})$. To avoid a cluttered notation, we will omit the hats over operators, so that all equations will have the same form as previously but considered as equations in $\mathrm{End}(\underline{V}\otimes\mathcal{H})$. The main advantage of this prescription will arise from the fact that, unlike $\mathbb{R}^{2n}$, the noncommutative space $\mathbb{R}^{2n}_\theta$ has a non-trivial K-theory which allows for gauge field configurations of non-trivial topological charge while retaining the simple geometry of flat contractible space.

3. Quiver gauge theory and graded connections

In this section we will exploit the fact that the G-equivariant reduction carried out in the previous section has a natural interpretation as the representation of a particular class of quivers in the category of vector bundles over the Kähler manifold M, i.e. as a quiver bundle over M [14, 15, 20]. The most natural notion of gauge field on a quiver bundle is provided by that of a *graded connection* as introduced in [18]. After describing some general aspects of the quivers related to our analysis, we will rewrite the equivariant decomposition of the

JHEP09(2006)054

gauge fields of the previous section in terms of graded connections on the pertinent quivers. Besides its mathematical elegance, the main advantage of this representation is that it will make the physical interpretations of our field configurations completely transparent later on. Treatments of the theory of quivers can be found in [21].

3.1 The $\mathrm{A}_{m_1+1} \oplus \mathrm{A}_{m_2+1}$ quiver and its representations

A quiver is an oriented graph, i.e. a set of vertices together with a set of arrows between the vertices. For a given pair of positive integers m_1, m_2, it is clear that the bundle diagram (2.12) can be naturally associated to a quiver $\mathsf{Q}_{(m_1,m_2)}$. The nodes of this quiver are labelled by monopole charges giving the vertex set $\mathsf{Q}^{(0)}_{(m_1,m_2)} = \{(v_i^{(1)}, v_\alpha^{(2)}) = (m_1 - 2i, m_2 - 2\alpha) \mid 0 \leq i \leq m_1\,,\, 0 \leq \alpha \leq m_2\}$. The arrow set is given by $\mathsf{Q}^{(1)}_{(m_1,m_2)} = \{\zeta^{(\ell)}_{i\alpha} \mid \ell = 1,2\,,\, 0 \leq i \leq m_1\,,\, 0 \leq \alpha \leq m_2\}$ with $\zeta^{(1)}_{i+1\,\alpha} : (v^{(1)}_{i+1}, v^{(2)}_\alpha) \mapsto (v^{(1)}_i, v^{(2)}_\alpha)$ and $\zeta^{(2)}_{i\,\alpha+1} : (v^{(1)}_i, v^{(2)}_{\alpha+1}) \mapsto (v^{(1)}_i, v^{(2)}_\alpha)$. A path in $\mathsf{Q}_{(m_1,m_2)}$ is a sequence of arrows in $\mathsf{Q}^{(1)}_{(m_1,m_2)}$ which compose. If the head of $\zeta^{(\ell)}_{i\alpha}$ is the same node as the tail of $\zeta^{(\ell')}_{i'\alpha'}$, then we may produce a path $\zeta^{(\ell')}_{i'\alpha'}\,\zeta^{(\ell)}_{i\alpha}$ consisting of $\zeta^{(\ell)}_{i\alpha}$ followed by $\zeta^{(\ell')}_{i'\alpha'}$. To each vertex $(m_1 - 2i, m_2 - 2\alpha)$ we associate the trivial path $e_{i\alpha}$ of length 0. Each arrow $\zeta^{(\ell)}_{i\alpha}$ itself may be associated to a path of length 1. A relation r of the quiver is a formal finite sum of paths. From (2.13) it follows that the set $\mathsf{R}_{(m_1,m_2)}$ of relations of $\mathsf{Q}_{(m_1,m_2)}$ are given by $r_{i\alpha} = \zeta^{(1)}_{i+1\,\alpha}\,\zeta^{(2)}_{i+1\,\alpha+1} - \zeta^{(2)}_{i\,\alpha+1}\,\zeta^{(1)}_{i+1\,\alpha+1}$ for $0 \leq i \leq m_1$, $0 \leq \alpha \leq m_2$.

If we set $M =$ point in the construction of section 2.2, then we obtain a representation $\underline{\mathcal{V}}$ of the quiver $\mathsf{Q}_{(m_1,m_2)}$ obtained by placing the G-modules $\underline{V}_{k_{i\alpha}}$ in (2.6) at the vertices $(m_1 - 2i, m_2 - 2\alpha)$. Recalling that the nodes of the quiver arose as the set of weights for the action of the Borel subgroup K on the bundle $E \to M$, we obtain natural equivalence functors between the categories of holomorphic representations of K and indecomposable representations of the quiver with relations $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$, and also with the category of holomorphic homogeneous vector bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1 \cong G^{\mathrm{c}}/K$. In particular, there is a one-to-one correspondence between G-equivariant vector bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ and commutative diagrams on the quiver $\mathsf{Q}_{(m_1,m_2)}$. In the case of a generic Kähler manifold M, any G-equivariant bundle over $M \times \mathbb{C}P^1 \times \mathbb{C}P^1$ defines a quiver representation obtained by placing the vector bundles $E_{k_{i\alpha}} \to M$ at the vertices $(m_1 - 2i, m_2 - 2\alpha)$, as in (2.12). It follows that there is a one-to-one correspondence between such bundles and indecomposable $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$-bundles over M. Neither the holomorphicity of the quiver representation nor the relations need generically hold for the decomposition of gauge fields given in section 2.3, but instead will arise as a dynamical effect from a specific choice of ansatz. Note that when one passes to the corresponding noncommutative gauge theory, one is faced with infinite-dimensional quiver representations $\underline{\mathcal{V}} \otimes \mathcal{H}$, and one of the goals of our later constructions will be to find appropriate truncations to finite-dimensional modules over $\mathsf{Q}_{(m_1,m_2)}$.

To aid in the construction of quiver representations, one defines the path algebra $\mathsf{A}_{(m_1,m_2)}$ of $\mathsf{Q}_{(m_1,m_2)}$ to be the vector space over $\mathbb{C}$ generated by all paths, together with the multiplication given by concatenation of paths. If two paths do not compose then their

product is defined to be 0. The trivial paths are idempotents, $e_{i\alpha}^2 = e_{i\alpha}$, and thereby define a collection of projectors on the finite-dimensional free algebra $\mathsf{A}_{(m_1,m_2)}$. Imposing relations on the quiver then amounts to taking the quotient of $\mathsf{A}_{(m_1,m_2)}$ by the ideal generated by the $r_{i\alpha}$. Given a representation $\underline{\mathcal{V}}$ of the algebra $\mathsf{A}_{(m_1,m_2)}$, we can form the vector spaces $\underline{V}_{k_{i\alpha}} = \underline{\mathcal{V}} \cdot e_{i\alpha} \cong \mathbb{C}^{k_{i\alpha}}$. The elements of $\mathsf{A}_{(m_1,m_2)}$ corresponding to arrows in $\mathsf{Q}_{(m_1,m_2)}$ yield linear maps between the $\underline{V}_{k_{i\alpha}}$ which have to satisfy the relations $r_{i\alpha} = 0$. It follows that representations of the path algebra $\mathsf{A}_{(m_1,m_2)}/\mathsf{R}_{(m_1,m_2)}$ are equivalent to quiver representations of $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$ [21]. Such a representation is specified by giving the ordered collection of positive integers $\vec{k} = \vec{k}_{\underline{\mathcal{V}}} := (k_{i\alpha})_{0\le i\le m_1, 0\le\alpha\le m_2}$, called the dimension vector of the quiver representation, at the vertices of $\mathsf{Q}_{(m_1,m_2)}$.

A useful set of quiver representations $\underline{\mathcal{P}}_{i\alpha}$ is defined for each vertex of $\mathsf{Q}_{(m_1,m_2)}$ by $\underline{\mathcal{P}}_{i\alpha} := e_{i\alpha} \cdot \mathsf{A}_{(m_1,m_2)}$, which is the subspace of $\mathsf{A}_{(m_1,m_2)}$ generated by all paths starting at the node $(m_1 - 2i, m_2 - 2\alpha)$. Multiplying on the right by elements of the path algebra $\mathsf{A}_{(m_1,m_2)}$ makes $\underline{\mathcal{P}}_{i\alpha}$ into a right $\mathsf{A}_{(m_1,m_2)}$-module and hence a quiver representation. This path algebra representation has many special properties. The collection of all modules $\underline{\mathcal{P}}_{i\alpha}$, $0 \le i \le m_1$, $0 \le \alpha \le m_2$ are exactly the set of all indecomposable projective representations of the quiver $\mathsf{Q}_{(m_1,m_2)}$, with the natural isomorphism

$$\mathsf{A}_{(m_1,m_2)} = \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \underline{\mathcal{P}}_{i\alpha} \tag{3.1}$$

as right $\mathsf{A}_{(m_1,m_2)}$-modules. Furthermore, for any quiver representation (2.6) there is a natural isomorphism

$$\mathrm{Hom}\big(\underline{\mathcal{P}}_{i\alpha}\,,\,\underline{\mathcal{V}}\big) = \underline{V}_{k_{i\alpha}}\,, \tag{3.2}$$

and in particular

$$\mathrm{Hom}\big(\underline{\mathcal{P}}_{j\beta}\,,\,\underline{\mathcal{P}}_{i\alpha}\big) \;=\; \big(\underline{\mathcal{P}}_{i\alpha}\big)_{j\beta} \;=\; e_{i\alpha} \cdot \mathsf{A}_{(m_1,m_2)} \cdot e_{j\beta} \tag{3.3}$$

is the vector space spanned by all paths from vertex $(v_i^{(1)}, v_\alpha^{(2)})$ to vertex $(v_j^{(1)}, v_\beta^{(2)})$. Imposing the relations $r_{i\alpha}$ identifies all such paths and one has $(\underline{\mathcal{P}}_{i\alpha})_{j\beta} \cong \mathbb{C}$ for the corresponding quiver representation of $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$.

A morphism $\underline{f} : \underline{\mathcal{V}} \to \underline{\mathcal{V}'}$ of two quiver representations is given by linear maps $f_{i\alpha} : \underline{V}_{k_{i\alpha}} \to \underline{V'}_{k'_{i\alpha}}$ for each vertex such that $\phi'^{(1)}_{i+1\,\alpha}\, f_{i\alpha} = f_{i+1\,\alpha}\, \phi^{(1)}_{i+1\,\alpha}$ and $\phi'^{(2)}_{i\,\alpha+1}\, f_{i\alpha} = f_{i\,\alpha+1}\, \phi^{(2)}_{i\,\alpha+1}$. This notion defines the abelian category of quiver representations (or equivalently of right $\mathsf{A}_{(m_1,m_2)}$-modules). If all linear maps $f_{i\alpha}$ are invertible, then $\underline{f}$ is called an isomorphism of quiver representations. Any two isomorphic representations necessarily have the same dimension vector $\vec{k}$. This provides a natural notion of gauge symmetry in quiver gauge theory. We will return to the issue of equivalence of representations of the quiver $\mathsf{Q}_{(m_1,m_2)}$ in section 4.5.

3.2 Matrix presentation of equivariant gauge fields

A convenient way of combining the reductions of equivariant gauge fields is through the formalism of graded connections introduced in [18]. The first step in this procedure is

to rewrite the decompositions of section 2.3 in a particular matrix form that reflects the representations of the path algebra given in (3.1)–(3.3). The basic idea is that, given the isomorphisms $(\underline{\mathcal{P}}_{i\alpha})_{j\beta} \cong \mathbb{C}$, one can identify (3.1) with an algebra of upper triangular complex matrices. For this, let us write the rank k equivariant bundle $E \to M$ in the $\mathbb{Z}_{m_1+1}\times\mathbb{Z}_{m_2+1}$-graded form

$$E \;:=\; \bigoplus_{i=0}^{m_1}\bigoplus_{\alpha=0}^{m_2} E_{k_{i\alpha}} \;=\; \bigoplus_{\alpha=0}^{m_2} E_{(m_1)\alpha} \qquad \text{with} \quad E_{(m_1)\alpha} \;:=\; \bigoplus_{i=0}^{m_1} E_{k_{i\alpha}}\ . \tag{3.4}$$

The algebra $\Omega_\sharp(M,E)$ of differential forms on the manifold M with values in the bundle E has a total $\mathbb{Z}_{m_1+1}\times\mathbb{Z}_{m_2+1}$ grading defined by combining the grading in (3.4) with the $\mathbb{Z}$-grading by form degree. Similarly, the $\mathbb{Z}_{m_1+1}\times\mathbb{Z}_{m_2+1}$ grading of the endomorphism bundle

$$\begin{aligned}\mathrm{End}(E) \;&=\; \bigoplus_{i,j=0}^{m_1}\bigoplus_{\alpha,\beta=0}^{m_2} \mathrm{Hom}(E_{k_{i\alpha}},E_{k_{j\beta}}) \\ &=\; \bigoplus_{\alpha=0}^{m_2} \mathrm{End}(E_{(m_1)\alpha}) \;\oplus\; \bigoplus_{\substack{\alpha,\beta=0\\ \alpha\neq\beta}}^{m_2} \mathrm{Hom}(E_{(m_1)\alpha},E_{(m_1)\beta})\end{aligned} \tag{3.5}$$

induces a total $\mathbb{Z}_{m_1+1}\times\mathbb{Z}_{m_2+1}$ grading on the endomorphism algebra $\Omega_\sharp(M,\mathrm{End}\ E)$.

A graded connection on E is a derivation on $\Omega_\sharp(M,E)$ which shifts the total $\mathbb{Z}_{m_1+1}\times\mathbb{Z}_{m_2+1}$ grading by 1, and is thus an element of the degree 1 subspace of $\Omega_\sharp(M,\mathrm{End}\ E)$. For a given module (2.6) over the quiver $\mathsf{Q}_{(m_1,m_2)}$, the zero-form components in this subspace represent the arrows of $\mathsf{Q}_{(m_1,m_2)}$ and are defined by appropriately assembling the Higgs fields of the equivariant gauge potentials into off-diagonal operators in (3.5) acting on the decomposition in (3.4). To this end we introduce square matrices of morphisms acting on the bundles $E_{(m_1)\alpha}$ through

$$\boldsymbol{\phi}^{(1)}_{(m_1)\alpha} \;:=\; \begin{pmatrix} 0 & \phi^{(1)}_{1\alpha} & 0 & \dots & 0 \\ 0 & 0 & \phi^{(1)}_{2\alpha} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \phi^{(1)}_{m_1\alpha} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad \text{with} \quad \alpha \;=\; 0,1,\dots,m_2 \tag{3.6}$$

and assemble them into a $k\times k$ matrix with respect to the grading (3.4) and (3.5) as

$$\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)} := \begin{pmatrix} \boldsymbol{\phi}^{(1)}_{(m_1)0} & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{\phi}^{(1)}_{(m_1)1} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{\phi}^{(1)}_{(m_1)2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \boldsymbol{\phi}^{(1)}_{(m_1)m_2} \end{pmatrix}\ . \tag{3.7}$$

Remembering that $\phi^{(2)}_{i\,m_2+1} := 0 \quad \forall i = 0,1,\ldots,m_1$, we similarly define matrices of morphisms on $E_{(m_1)\,\alpha+1}$ through

$$\boldsymbol{\phi}^{(2)}_{(m_1)\,\alpha+1} := \begin{pmatrix} \phi^{(2)}_{0\,\alpha+1} & 0 & 0 & \ldots & 0 \\ 0 & \phi^{(2)}_{1\,\alpha+1} & 0 & \ldots & 0 \\ 0 & 0 & \phi^{(2)}_{2\,\alpha+1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & \phi^{(2)}_{m_1\,\alpha+1} \end{pmatrix} \qquad \text{with} \quad \alpha = 0,1,\ldots,m_2 \tag{3.8}$$

and assemble them into a $k\times k$ matrix acting on (3.4) as

$$\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)} := \begin{pmatrix} 0 & \boldsymbol{\phi}^{(2)}_{(m_1)1} & 0 & \ldots & 0 \\ 0 & 0 & \boldsymbol{\phi}^{(2)}_{(m_1)2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & \boldsymbol{\phi}^{(2)}_{(m_1)m_2} \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} . \tag{3.9}$$

The finite dimensionality of the path algebra (3.1) corresponds to the generic nilpotency properties

$$\begin{aligned} &\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}, \Big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\Big)^2, \ldots, \Big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\Big)^{m_1} \neq 0 \qquad \text{but} \qquad \Big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\Big)^{m_1+1} = 0\,, \\ &\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}, \Big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\Big)^2, \ldots, \Big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\Big)^{m_2} \neq 0 \qquad \text{but} \qquad \Big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\Big)^{m_2+1} = 0\,. \end{aligned} \tag{3.10}$$

The holomorphic relations (2.13) now take the simple algebraic form of commutativity of the matrices (3.7) and (3.9) as

$$\Big[\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\,,\,\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\Big] = 0\,. \tag{3.11}$$

Although a very natural requirement, the condition (3.11) is not necessary for the present formulation and the relations $\mathsf{R}_{(m_1,m_2)}$ of the quiver $\mathsf{Q}_{(m_1,m_2)}$ will only play a prominent role in the subsequent sections.

The one-form components of the graded connection represent the vertices of $\mathsf{Q}_{(m_1,m_2)}$ and correspond to diagonal operators in the decomposition (3.5). They can be written using the canonical orthogonal projections $\Pi_{i\alpha} : E \to E_{k_{i\alpha}}$ of rank 1 obeying

$$\Pi_{i\alpha}\,\Pi_{j\beta} = \delta_{ij}\,\delta_{\alpha\beta}\,\Pi_{i\alpha} \tag{3.12}$$

which may be represented, with respect to the decomposition (3.4), by the diagonal matrices

$$\Pi_{i\alpha} = \big(\delta_{ij}\,\delta_{il}\,\delta_{\alpha\beta}\,\delta_{\alpha\gamma}\big)^{j,l=0,1,\ldots,m_1}_{\beta,\gamma=0,1,\ldots,m_2}\,. \tag{3.13}$$

The gauge potentials living at the vertices of the quiver may then be assembled into the $k\times k$ matrix

$$\boldsymbol{A}^{(m_1,\,m_2)} := \sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2} A^{i\alpha}\otimes\Pi_{i\alpha}\,. \tag{3.14}$$

To rewrite the equivariant decomposition of the components of the gauge potentials on the bundle $\mathcal{E} \to M \times \mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$, we assemble the monopole connections into the matrices

$$\boldsymbol{a}^{(m_1)} := \sum_{i=0}^{m_1} a^{(1)}_{m_1-2i} \otimes \Pi_i \qquad \text{with} \quad \Pi_i := \bigoplus_{\alpha=0}^{m_2} \Pi_{i\alpha}\,, \tag{3.15}$$

$$\boldsymbol{a}^{(m_2)} := \sum_{\alpha=0}^{m_2} a^{(2)}_{m_2-2\alpha} \otimes \Pi_\alpha \qquad \text{with} \quad \Pi_\alpha := \bigoplus_{i=0}^{m_1} \Pi_{i\alpha} \tag{3.16}$$

and the monopole charges labelling the vertices of $\mathsf{Q}_{(m_1,m_2)}$ into the matrices

$$\boldsymbol{\Upsilon}^{(1)}_{(m_1,\,m_2)} := \sum_{i=0}^{m_1} (m_1-2i)\,\Pi_i\,, \tag{3.17}$$

$$\boldsymbol{\Upsilon}^{(2)}_{(m_1,\,m_2)} := \sum_{\alpha=0}^{m_2} (m_2-2\alpha)\,\Pi_\alpha\ . \tag{3.18}$$

Then the ansatz (2.22)–(2.27) can be rewritten in terms of the matrix operators (3.6)–(3.9) and (3.14)–(3.16) as

$$\mathcal{A}_\mu = \big(\boldsymbol{A}^{(m_1,m_2)}\big)_\mu \otimes 1 \otimes 1\,, \tag{3.19}$$

$$\mathcal{A}_{y_1} = \mathbf{1}_k \otimes \big(\boldsymbol{a}^{(m_1)}\big)_{y_1} \otimes 1 - \big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\big)^\dagger \otimes (\beta_1)_{y_1} \otimes 1\,, \tag{3.20}$$

$$\mathcal{A}_{y_2} = \mathbf{1}_k \otimes 1 \otimes \big(\boldsymbol{a}^{(m_2)}\big)_{y_2} - \big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\big)^\dagger \otimes 1 \otimes (\beta_2)_{y_2}\,, \tag{3.21}$$

$$\mathcal{A}_{\bar y_1} = \mathbf{1}_k \otimes \big(\boldsymbol{a}^{(m_1)}\big)_{\bar y_1} \otimes 1 + \boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)} \otimes (\bar\beta_1)_{\bar y_1} \otimes 1\,, \tag{3.22}$$

$$\mathcal{A}_{\bar y_2} = \mathbf{1}_k \otimes 1 \otimes \big(\boldsymbol{a}^{(m_2)}\big)_{\bar y_2} + \boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)} \otimes 1 \otimes (\bar\beta_2)_{\bar y_2}\ . \tag{3.23}$$

As we will see in section 3.4, the scalar potential in (2.50) can be rewritten entirely in terms of the natural algebraic operators $\boldsymbol{\Upsilon}^{(1)}_{(m_1,\,m_2)} - \big[\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}, (\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)})^\dagger\big]$, $\boldsymbol{\Upsilon}^{(2)}_{(m_1,\,m_2)} - \big[\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}, (\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)})^\dagger\big]$, $\big[\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}, \boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\big]$ and $\big[\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}, (\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)})^\dagger\big]$ on the quiver $\mathsf{Q}_{(m_1,m_2)}$.

3.3 Examples

To help understand the forms of the matrix presentations introduced above, it is instructive to look at some explicit examples of $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$-bundles over M before proceeding further with more of the general formalism.

$\boldsymbol{(m_1,m_2)=(m,0)}$. In this case the vertical arrows $\zeta^{(2)}_{i\alpha}$ of the quiver $\mathsf{Q}_{(m,0)}$ are all 0 and the quiver bundle (2.12) collapses to the holomorphic *chain* [13]

$$E_{k_{m\,0}} \xrightarrow{\ \phi^{(1)}_{m\,0}\ } E_{k_{m-1\,0}} \xrightarrow{\ \phi^{(1)}_{m-1\,0}\ } \cdots \xrightarrow{\ \phi^{(1)}_{10}\ } E_{k_{00}} \tag{3.24}$$

considered in [18]. The quiver $\mathsf{Q}_{(m,0)}$ is called the A_{m+1}-quiver. The set of relations $\mathsf{R}_{(m,0)}$ is empty and the non-vanishing Higgs fields are assembled into the zero-form graded

connection component

$$\boldsymbol{\phi}^{(1)}_{(m,0)} \;=\; \boldsymbol{\phi}^{(1)}_{(m)0} \;=\; \begin{pmatrix} 0 & \phi^{(1)}_{10} & 0 & \dots & 0 \\ 0 & 0 & \phi^{(1)}_{20} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \phi^{(1)}_{m0} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad \text{on} \quad E \;=\; E_{(m)0} \;=\; \bigoplus_{i=0}^{m} E_{k_{i0}} \;. \tag{3.25}$$

The simplest case $m = 1$ gives a holomorphic *triple* [12] and corresponds to the more standard superconnections, having $(\boldsymbol{\phi}^{(1)}_{(1,0)})^2 = 0$, which characterize the low-energy field content on brane-antibrane systems with the tachyon field $\phi^{(1)}_{10}$ between the branes and antibranes [2, 17]. A completely analogous characterization holds for the charge configuration $(m_1, m_2) = (0, m)$. As we will discuss further in the subsequent sections, for generic m_1, m_2 the set of relations $\mathsf{R}_{(m_1,m_2)}$, making the vector space $(\underline{\mathcal{P}}_{i\alpha})_{j\beta}$ one-dimensional, implies that the quiver $\mathsf{Q}_{(m_1,m_2)}$ can always be naturally mapped (e.g. via a lexicographic ordering) onto an A_{m+1}-quiver. This will become evident from the other examples considered below, and will have important physical ramifications later on.

$\boldsymbol{(m_1, m_2) = (1, 1)}$. In this case the quiver bundle truncates to a square

$$\begin{array}{ccc} E_{k_{10}} & \xrightarrow{\;\phi^{(1)}_{10}\;} & E_{k_{00}} \\ {\scriptstyle\phi^{(2)}_{11}}\uparrow & & \uparrow{\scriptstyle\phi^{(2)}_{01}} \\ E_{k_{11}} & \xrightarrow[\;\phi^{(1)}_{11}\;]{} & E_{k_{01}} \end{array} \tag{3.26}$$

and uniqueness of the bundle morphism on $E_{k_{11}} \to E_{k_{00}}$ (or of the corresponding path in the path algebra $\mathsf{A}_{(1,1)}$) yields the single holomorphic relation

$$\phi^{(2)}_{01}\,\phi^{(1)}_{11} = \phi^{(1)}_{10}\,\phi^{(2)}_{11} \;. \tag{3.27}$$

The equivariant graded connection admits the matrix presentation

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^{00,00} & \phi^{(1)}_{10} & \phi^{(2)}_{01} & 0 \\ -\big(\phi^{(1)}_{10}\big)^\dagger & \mathcal{A}^{10,10} & 0 & \phi^{(2)}_{11} \\ -\big(\phi^{(2)}_{11}\big)^\dagger & 0 & \mathcal{A}^{01,01} & \phi^{(1)}_{11} \\ 0 & -\big(\phi^{(2)}_{11}\big)^\dagger & -\big(\phi^{(1)}_{11}\big)^\dagger & \mathcal{A}^{11,11} \end{pmatrix} \;. \tag{3.28}$$

$\boldsymbol{(m_1, m_2) = (2, 1)}$. The quiver bundle over M associated to $\mathsf{Q}_{(2,1)}$ is given by

$$\begin{array}{ccccc} E_{k_{20}} & \xrightarrow{\;\phi^{(1)}_{20}\;} & E_{k_{10}} & \xrightarrow{\;\phi^{(1)}_{10}\;} & E_{k_{00}} \\ {\scriptstyle\phi^{(2)}_{21}}\uparrow & & {\scriptstyle\phi^{(2)}_{11}}\uparrow & & \uparrow{\scriptstyle\phi^{(2)}_{01}} \\ E_{k_{21}} & \xrightarrow[\;\phi^{(1)}_{21}\;]{} & E_{k_{11}} & \xrightarrow[\;\phi^{(1)}_{11}\;]{} & E_{k_{01}} \end{array} \tag{3.29}$$

with the pair of holomorphic relations

$$\phi_{11}^{(2)}\,\phi_{21}^{(1)} \;=\; \phi_{20}^{(1)}\,\phi_{21}^{(2)} \qquad \text{and} \qquad \phi_{01}^{(2)}\,\phi_{11}^{(1)} \;=\; \phi_{10}^{(1)}\,\phi_{11}^{(2)}\;. \tag{3.30}$$

The graded connection zero-form components

$$\boldsymbol{\phi}_{(2,1)}^{(1)} \;:=\; \begin{pmatrix} 0 & \phi_{10}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_{20}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{11}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\phi}_{(2,1)}^{(2)} \;:=\; \begin{pmatrix} 0 & 0 & 0 & \phi_{01}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{11}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{21}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.31}$$

satisfy the nilpotent relations

$$\big(\boldsymbol{\phi}_{(2,1)}^{(1)}\big)^2 \;\neq\; 0\,, \quad \big(\boldsymbol{\phi}_{(2,1)}^{(1)}\big)^3 \;=\; 0 \quad \text{and} \quad \big(\boldsymbol{\phi}_{(2,1)}^{(2)}\big)^2 \;=\; 0\;. \tag{3.32}$$

It is straightforward to check that the holomorphic relations (3.30) follow from the commutativity condition (3.11) in this case.

$\boldsymbol{(m_1, m_2) = (2, 2)}$. Finally, the $(\mathsf{Q}_{(2,2)}\,,\,\mathsf{R}_{(2,2)})$-bundle is given by

$$\begin{array}{ccccc} E_{k_{20}} & \xrightarrow[]{\phi_{20}^{(1)}} & E_{k_{10}} & \xrightarrow[]{\phi_{10}^{(1)}} & E_{k_{00}} \\ {\scriptstyle \phi_{21}^{(2)}}\uparrow & & {\scriptstyle \phi_{11}^{(2)}}\uparrow & & \uparrow{\scriptstyle \phi_{01}^{(2)}} \\ E_{k_{21}} & \xrightarrow[\phi_{21}^{(1)}]{} & E_{k_{11}} & \xrightarrow[\phi_{11}^{(1)}]{} & E_{k_{01}} \\ {\scriptstyle \phi_{22}^{(2)}}\uparrow & & {\scriptstyle \phi_{12}^{(2)}}\uparrow & & \uparrow{\scriptstyle \phi_{02}^{(2)}} \\ E_{k_{22}} & \xrightarrow[\phi_{22}^{(1)}]{} & E_{k_{12}} & \xrightarrow[\phi_{12}^{(1)}]{} & E_{k_{02}} \end{array} \tag{3.33}$$

with

$$\boldsymbol{\phi}_{(2,2)}^{(1)} \oplus \boldsymbol{\phi}_{(2,2)}^{(2)} = \begin{pmatrix} 0 & \phi_{10}^{(1)} & 0 & \phi_{01}^{(2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_{20}^{(1)} & 0 & \phi_{11}^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{21}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{11}^{(1)} & 0 & \phi_{02}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{21}^{(1)} & 0 & \phi_{12}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{22}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{12}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{22}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.34}$$

satisfying

$$\big(\boldsymbol{\phi}_{(2,2)}^{(\ell)}\big)^2 \;\neq\; 0 \qquad \text{and} \qquad \big(\boldsymbol{\phi}_{(2,2)}^{(\ell)}\big)^3 \;=\; 0 \qquad \text{for} \quad \ell \;=\; 1, 2\;. \tag{3.35}$$

3.4 Graded connections on $\mathsf{Q}_{(m_1,m_2)}$

We would now like to write the graded connections as intrinsic objects to the quiver bundle (2.12) over M, without explicit reference to their origin as connections on the equivariant gauge bundle $\mathcal{E} \to M \times \mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$. For this, we will introduce a more direct dimensional reduction of the gauge potential $\mathcal{A}$. The construction exploits the usual canonical isomorphism between the complexified exterior algebra bundle over $M \times \mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$ and the corresponding graded Clifford algebra bundle, which sends the exterior product into completely antisymmetrized Clifford multiplication and the local cotangent basis $\mathrm{d}x^{\hat{\mu}}$ onto the Clifford algebra generators $\Gamma^{\hat{\mu}}$ obeying the anticommutation relations

$$\Gamma^{\hat{\mu}}\,\Gamma^{\hat{\nu}} + \Gamma^{\hat{\nu}}\,\Gamma^{\hat{\mu}} \;=\; -2\,g^{\hat{\mu}\hat{\nu}}\;\mathbf{1}_{2^{n+2}} \qquad \text{with} \quad \hat{\mu},\hat{\nu} \;=\; 1,\dots,2n+4\;. \tag{3.36}$$

The gamma-matrices in (3.36) may be decomposed as

$$\left\{\Gamma^{\hat{\mu}}\right\} \;=\; \left\{\Gamma^{\mu},\,\Gamma^{y_1},\,\Gamma^{\bar{y}_1},\,\Gamma^{y_2},\,\Gamma^{\bar{y}_2}\right\} \tag{3.37}$$

where

$$\Gamma^{\mu} = \gamma^{\mu} \otimes \mathbf{1}_2 \otimes \mathbf{1}_2\,, \tag{3.38}$$

and $\gamma^{\mu} = -(\gamma^{\mu})^{\dagger}$ are the $2^n \times 2^n$ matrices which locally generate the Clifford algebra bundle over M and which obey the anticommutation relations

$$\gamma^{\mu}\,\gamma^{\nu} + \gamma^{\nu}\,\gamma^{\mu} \;=\; -2\,g^{\mu\nu}\;\mathbf{1}_{2^n} \qquad \text{with} \quad \mu,\nu \;=\; 1,\dots,2n\;. \tag{3.39}$$

The spherical components are given by

$$\Gamma^{y_1} = \gamma \otimes \gamma^{y_1} \otimes \mathbf{1}_2\,, \quad \Gamma^{\bar{y}_1} \;=\; \gamma \otimes \gamma^{\bar{y}_1} \otimes \mathbf{1}_2\,, \tag{3.40}$$

$$\Gamma^{y_2} = \gamma \otimes \sigma_3 \otimes \gamma^{y_2}\,, \quad \Gamma^{\bar{y}_2} \;=\; \gamma \otimes \sigma_3 \otimes \gamma^{\bar{y}_2}\,, \tag{3.41}$$

where

$$\gamma^{y_\ell} \;=\; -\frac{1}{R_\ell}\,(1 + y_\ell \bar{y}_\ell)\;\sigma_+ \qquad \text{and} \qquad \gamma^{\bar{y}_\ell} \;=\; \frac{1}{R_\ell}\,(1 + y_\ell \bar{y}_\ell)\;\sigma_- \tag{3.42}$$

are the Clifford algebra generators over $\mathbb{C}P^1_{(\ell)}$ for $\ell = 1,2$, with the constant $\mathrm{sl}(2,\mathbb{C})$ generators given by (2.9), (2.10). The chirality operator over M is

$$\gamma \;=\; \frac{\mathrm{i}^{\,n}}{(2n)!\,\sqrt{g}}\;\epsilon_{\mu_1\cdots\mu_{2n}}\,\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}} \qquad \text{with} \qquad (\gamma)^2 \;=\; \mathbf{1}_{2^n} \quad \text{and} \quad \gamma\,\gamma^{\mu} \;=\; -\gamma^{\mu}\,\gamma\;. \tag{3.43}$$

With this set-up we may now write the equivariant gauge potential given by (2.22)–(2.27) as the graded connection

$$\begin{aligned}
\hat{\mathcal{A}} \,:=\, \Gamma^{\hat{\mu}}\,\mathcal{A}_{\hat{\mu}} \;&=\; \Gamma^{\mu}\,\mathcal{A}_{\mu} + \Gamma^{y_1}\,\mathcal{A}_{y_1} + \Gamma^{\bar{y}_1}\,\mathcal{A}_{\bar{y}_1} + \Gamma^{y_2}\,\mathcal{A}_{y_2} + \Gamma^{\bar{y}_2}\,\mathcal{A}_{\bar{y}_2} \\
&=\; \gamma^{\mu}\,\big(\boldsymbol{A}^{(m_1,\,m_2)}\big)_{\mu} \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{R_1}\,\big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\big)\,\gamma \otimes \sigma_- \otimes \mathbf{1}_2 + \frac{1}{R_1}\,\big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\big)^{\dagger}\,\gamma \otimes \sigma_+ \otimes \mathbf{1}_2 \\
&\quad + \frac{1}{R_2}\,\big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\big)\,\gamma \otimes \sigma_3 \otimes \sigma_- + \frac{1}{R_2}\,\big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\big)^{\dagger}\,\gamma \otimes \sigma_3 \otimes \sigma_+ \\
&\quad + \gamma \otimes \Big(\gamma^{\bar{y}_1}\,\big(\boldsymbol{a}^{(m_1)}\big)_{\bar{y}_1} + \gamma^{y_1}\,\big(\boldsymbol{a}^{(m_1)}\big)_{y_1}\Big) \otimes \mathbf{1}_2 + \gamma \otimes \sigma_3 \otimes \Big(\gamma^{\bar{y}_2}\,\big(\boldsymbol{a}^{(m_2)}\big)_{\bar{y}_2} + \gamma^{y_2}\,\big(\boldsymbol{a}^{(m_2)}\big)_{y_2}\Big)\;,
\end{aligned} \tag{3.44}$$

where

$$\gamma^{\bar{y}_\ell}\left(\boldsymbol{a}^{(m_\ell)}\right)_{\bar{y}_\ell} + \gamma^{y_\ell}\left(\boldsymbol{a}^{(m_\ell)}\right)_{y_\ell} \;=\; \frac{1}{R_\ell}\,(1+y_\ell\bar{y}_\ell)\,\Big(\left(\boldsymbol{a}^{(m_\ell)}\right)_{\bar{y}_\ell}\sigma_- - \left(\boldsymbol{a}^{(m_\ell)}\right)_{y_\ell}\sigma_+\Big)$$
$$\text{for}\quad \ell \;=\; 1,2\;. \tag{3.45}$$

As desired, the zero-form components in (3.44) involving $\boldsymbol{\phi}^{(\ell)}_{(m_1,m_2)}$ are independent of the coordinates $(y_\ell,\bar{y}_\ell)\in\mathbb{C}P^1_{(\ell)}$ and they anticommute with the one-form components involving $\boldsymbol{A}^{(m_1,m_2)}$ due to their couplings with the chirality operator (3.43). From (2.41)–(2.47) the curvature of the graded connection (3.44) is found to be

$$\begin{aligned}
\hat{\mathcal{F}} \;&:=\; \tfrac{1}{4}\left[\Gamma^{\hat{\mu}}\,,\,\Gamma^{\hat{\nu}}\right]\mathcal{F}_{\hat{\mu}\hat{\nu}}\\
&=\; \tfrac{1}{4}\left[\gamma^{\mu}\,,\,\gamma^{\nu}\right]\left(\boldsymbol{F}^{(m_1,\,m_2)}\right)_{\mu\nu}\otimes\mathbf{1}_2\otimes\mathbf{1}_2\\
&\quad-\frac{1}{R_1}\,\gamma\left(\gamma^{\mu}\,D_\mu\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\right)\otimes\sigma_-\otimes\mathbf{1}_2+\frac{1}{R_1}\,\gamma\left(\gamma^{\mu}\,D_\mu\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\right)^\dagger\otimes\sigma_+\otimes\mathbf{1}_2\\
&\quad-\frac{1}{R_2}\,\gamma\left(\gamma^{\mu}\,D_\mu\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right)\otimes\sigma_3\otimes\sigma_-+\frac{1}{R_2}\,\gamma\left(\gamma^{\mu}\,D_\mu\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right)^\dagger\otimes\sigma_3\otimes\sigma_+\\
&\quad+\frac{1}{2\,R_1^2}\,\left(\boldsymbol{\Upsilon}^{(1)}_{(m_1,\,m_2)}-\left[\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\,,\,\left(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\right)^\dagger\right]\right)\,\mathbf{1}_{2^n}\otimes\sigma_3\otimes\mathbf{1}_2\\
&\quad+\frac{1}{2\,R_2^2}\,\left(\boldsymbol{\Upsilon}^{(2)}_{(m_1,\,m_2)}-\left[\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\,,\,\left(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right)^\dagger\right]\right)\,\mathbf{1}_{2^n}\otimes\mathbf{1}_2\otimes\sigma_3\\
&\quad+\frac{1}{R_1\,R_2}\,\left[\boldsymbol{\phi}^{(1)}_{(m_1,m_2)}\,,\,\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right]\,\mathbf{1}_{2^n}\otimes\sigma_-\otimes\sigma_-\\
&\quad+\frac{1}{R_1\,R_2}\,\left[\boldsymbol{\phi}^{(1)}_{(m_1,m_2)}\,,\,\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right]^\dagger\,\mathbf{1}_{2^n}\otimes\sigma_+\otimes\sigma_+\\
&\quad+\frac{1}{R_1\,R_2}\,\left[\boldsymbol{\phi}^{(1)}_{(m_1,m_2)}\,,\,\left(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right)^\dagger\right]\,\mathbf{1}_{2^n}\otimes\sigma_-\otimes\sigma_+\\
&\quad+\frac{1}{R_1\,R_2}\,\left[\boldsymbol{\phi}^{(1)}_{(m_1,m_2)}\,,\,\left(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\right)^\dagger\right]^\dagger\,\mathbf{1}_{2^n}\otimes\sigma_+\otimes\sigma_-
\end{aligned}\tag{3.46}$$

where $\boldsymbol{F}^{(m_1,\,m_2)}:=\mathrm{d}\boldsymbol{A}^{(m_1,\,m_2)}+\boldsymbol{A}^{(m_1,\,m_2)}\wedge\boldsymbol{A}^{(m_1,\,m_2)}=\frac{1}{2}\left(\boldsymbol{F}^{(m_1,\,m_2)}\right)_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$.

The graded curvature (3.46) is completely independent of the spherical coordinates. Using (3.46) and standard gamma-matrix trace formulas [18], it is possible to recast the dimensionally reduced Yang-Mills action functional (2.50) in the compact form

$$S_{\rm YM}=\frac{\pi^2\,R_1^2\,R_2^2}{2^n}\,\int_M\mathrm{d}^{2n}x\,\sqrt{g}\,\mathrm{tr}_{k\times k}\,\mathrm{Tr}_{\mathbb{C}^{2^{n+2}}}\,\hat{\mathcal{F}}^2\,,\tag{3.47}$$

where the trace $\mathrm{Tr}_{\mathbb{C}^{2^{n+2}}}$ is taken over the representation space of (3.36) and may be thought of as an "integral" over the Clifford algebra. Thus the entire equivariant gauge theory on $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ may be elegantly rewritten as an *ordinary* Yang-Mills gauge theory of *graded connections* on the corresponding *quiver bundle* over M.

4. Noncommutative instantons and quiver vortices

We will now proceed to the construction of explicit equivariant instanton solutions. We will build both BPS and non-BPS configurations of the Yang-Mills equations on the noncom-

mutative space $\mathbb{R}^{2n}_\theta \times \mathbb{C}P^1 \times \mathbb{C}P^1$. We then describe some general properties of the moduli space of noncommutative instantons in this instance.

4.1 BPS equations

The equations of motion which follow from varying the Yang-Mills lagrangian (2.48) on the Kähler manifold $M \times \mathbb{C}P^1 \times \mathbb{C}P^1$ are given by

$$\frac{1}{\sqrt{\hat{g}}}\,\partial_{\hat{\mu}}\big(\sqrt{\hat{g}}\;\mathcal{F}^{\hat{\mu}\hat{\nu}}\big) + \big[\mathcal{A}_{\hat{\mu}}\,,\,\mathcal{F}^{\hat{\mu}\hat{\nu}}\big] = 0\;. \tag{4.1}$$

The BPS configurations which satisfy (4.1) are provided by solutions of the DUY equations [11]

$$*\Omega \wedge \mathcal{F} \;=\; 0 \qquad \text{and} \qquad \mathcal{F}^{2,0} \;=\; 0 \;=\; \mathcal{F}^{0,2}\,, \tag{4.2}$$

where $*$ is the Hodge duality operator and $\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$ is the Kähler decomposition of the gauge field strength. In the local complex coordinates (z^a, y_1, y_2) these equations take the form

$$g^{a\bar{b}}\,\mathcal{F}_{z^a\bar{z}^{\bar{b}}} + g^{y_1\bar{y}_1}\,\mathcal{F}_{y_1\bar{y}_1} + g^{y_2\bar{y}_2}\,\mathcal{F}_{y_2\bar{y}_2} = 0\,, \tag{4.3}$$

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}} = 0\,, \tag{4.4}$$

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}_1} = 0 \;=\; \mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}_2}\,, \tag{4.5}$$

$$\mathcal{F}_{\bar{y}_1\bar{y}_2} = 0\,, \tag{4.6}$$

along with their complex conjugates for $a, b = 1, \dots, n$.

In terms of the equivariant decomposition (2.41)–(2.47), the DUY equations read

$$\begin{aligned} g^{a\bar{b}}\,F^{i\alpha}_{a\bar{b}} \;&=\; \frac{1}{2\,R_1^2}\,\Big[m_1 - 2i + \big(\phi^{(1)}_{i\alpha}\big)^\dagger\,\phi^{(1)}_{i\alpha} - \phi^{(1)}_{i+1\,\alpha}\,\big(\phi^{(1)}_{i+1\,\alpha}\big)^\dagger\Big] \\ &\quad + \frac{1}{2\,R_2^2}\,\Big[m_2 - 2\alpha + \big(\phi^{(2)}_{i\alpha}\big)^\dagger\,\phi^{(2)}_{i\alpha} - \phi^{(2)}_{i\,\alpha+1}\,\big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger\Big] \end{aligned} \tag{4.7}$$

and

$$F^{i\alpha}_{\bar{a}\bar{b}} = 0\,, \tag{4.8}$$

$$\partial_{\bar{a}}\phi^{(1)}_{i+1\,\alpha} + A^{i\alpha}_{\bar{a}}\,\phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha}\,A^{i+1\,\alpha}_{\bar{a}} = 0\,, \tag{4.9}$$

$$\partial_{\bar{a}}\phi^{(2)}_{i\,\alpha+1} + A^{i\alpha}_{\bar{a}}\,\phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,A^{i\,\alpha+1}_{\bar{a}} = 0\,, \tag{4.10}$$

$$\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1} = 0\,, \tag{4.11}$$

along with their complex conjugates. Eq. (4.7) gives hermitean conditions on the curvatures of $E_{k_{i\alpha}} \to M$, while (4.8) implies that $E_{k_{i\alpha}}$ are holomorphic vector bundles with connections $A^{i\alpha}$. The conditions (4.9) and (4.10) then mean that the bundle maps on the quiver bundle (2.12) are holomorphic. Eq. (4.11) imposes the relations $\mathsf{R}_{(m_1,m_2)}$ on the quiver bundle. Note that the analogous non-holomorphic relations, specified by the vanishing of (2.47), do not arise as BPS conditions.

The BPS energies may be computed by noting that the action functional (2.50) evaluated on equivariant connections $\mathcal{A}$ of the bundle $\mathcal{E} \to M \times \mathbb{C}P^1 \times \mathbb{C}P^1$ may be written as [15]

$$S_{\rm YM} = \frac{1}{4} \int_{M\times\mathbb{C}P^1\times\mathbb{C}P^1} {\rm d}^{2n+4}x \, \sqrt{\hat{g}} \, {\rm tr}_{k\times k}\big(\Omega^{\hat{\mu}\hat{\nu}} \, \mathcal{F}_{\hat{\mu}\hat{\nu}}\big)^2 - 2\pi^2 \, {\rm Ch}_2(\mathcal{E}) \,, \tag{4.12}$$

where

$${\rm Ch}_2(\mathcal{E}) = -\frac{1}{8\pi^2} \int_{M\times\mathbb{C}P^1\times\mathbb{C}P^1} \frac{\Omega^n}{n!} \wedge {\rm tr}_{k\times k} \, \mathcal{F} \wedge \mathcal{F} \tag{4.13}$$

is a Chern-Weil topological invariant of $\mathcal{E}$. Eq. (4.12) shows that the Yang-Mills action is bounded from below as $S_{\rm YM} \geq S_{\rm BPS} := -2\pi^2 \, {\rm Ch}_2(\mathcal{E})$, with equality precisely when the DUY equations (4.2) are satisfied. By substituting in (2.5) and the equivariant decomposition (2.41)–(2.47), after integration over $\mathbb{C}P^1\times\mathbb{C}P^1$ one finds

$$\begin{aligned} S_{\rm BPS} \;=\; & 2\pi^2 \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} \Big\{ {\rm vol}\, M \, \big[(m_1 - 2i) \, (m_2 - 2\alpha) \, k_{i\alpha} \\ & + 4 \, \big(R_2^2 \, (m_1 - 2i) + R_1^2 \, (m_2 - 2\alpha) \big) \, \deg \, E_{k_{i\alpha}} \big] - 64\pi^2 \, R_1^2 \, R_2^2 \, {\rm Ch}_2(E_{k_{i\alpha}}) \\ & + \int_M {\rm d}^{2n}x \, \sqrt{g} \, {\rm tr}_{k_{i\alpha}\times k_{i\alpha}} \Big[\big(\phi^{(1)}_{i+1\,\alpha+1}\big)^\dagger \, \big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger \, \phi^{(1)}_{i+1\,\alpha} \, \phi^{(2)}_{i+1\,\alpha+1} - \big(\phi^{(1)}_{i\alpha}\big)^\dagger \, \phi^{(1)}_{i\alpha} \, \big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger \, \phi^{(2)}_{i\,\alpha+1} \\ & + \big(\phi^{(1)}_{i+1\,\alpha}\big)^\dagger \, \phi^{(1)}_{i+1\,\alpha+1} \, \big(\phi^{(2)}_{i+1\,\alpha+1}\big)^\dagger \, \phi^{(2)}_{i\,\alpha+1} - \phi^{(1)}_{i+1\,\alpha} \, \big(\phi^{(1)}_{i+1\,\alpha}\big)^\dagger \, \big(\phi^{(2)}_{i\alpha}\big)^\dagger \, \phi^{(2)}_{i\alpha} \Big] \Big\} \,, \end{aligned} \tag{4.14}$$

where ${\rm vol}\, M = \int_M \omega^n/n!$ is the volume of the Kähler manifold M and

$$\deg \, E_{k_{i\alpha}} = \frac{\rm i}{{\rm vol}\, M} \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge {\rm tr}_{k_{i\alpha}\times k_{i\alpha}} \, F^{i\alpha} \tag{4.15}$$

is the degree of the rank $k_{i\alpha}$ bundle $E_{k_{i\alpha}} \to M$.

To cast these equations on the noncommutative space $M = \mathbb{R}^{2n}_\theta$, we introduce the operators

$$X^{i\alpha}_a \;:=\; A^{i\alpha}_a + \theta_{a\bar{b}} \, \bar{z}^{\bar{b}} \qquad \text{and} \qquad X^{i\alpha}_{\bar{a}} \;:=\; A^{i\alpha}_{\bar{a}} + \theta_{\bar{a}b} \, z^b \ . \tag{4.16}$$

In terms of these operators the antiholomorphic bi-fundamental covariant derivatives take the form

$$D_{\bar{a}}\phi^{(1)}_{i+1\,\alpha} \;=\; X^{i\alpha}_{\bar{a}} \, \phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha} \, X^{i+1\,\alpha}_{\bar{a}} \qquad \text{and} \qquad D_{\bar{a}}\phi^{(2)}_{i\,\alpha+1} \;=\; X^{i\alpha}_{\bar{a}} \, \phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1} \, X^{i\,\alpha+1}_{\bar{a}} \,, \tag{4.17}$$

while the components of the field strength tensor become

$$F^{i\alpha}_{a\bar{b}} \;=\; \big[X^{i\alpha}_a \,, \, X^{i\alpha}_{\bar{b}} \big] + \theta_{a\bar{b}} \,, \quad F^{i\alpha}_{\bar{a}\bar{b}} \;=\; \big[X^{i\alpha}_{\bar{a}} \,, \, X^{i\alpha}_{\bar{b}} \big] \quad \text{and} \quad F^{i\alpha}_{ab} \;=\; \big[X^{i\alpha}_a \,, \, X^{i\alpha}_b \big] \,. \tag{4.18}$$

The noncommutative DUY equations (without the complex conjugates) then read

$$\delta^{a\bar{b}}\left(\left[X_a^{i\alpha}\,,\,X_{\bar{b}}^{i\alpha}\right]+\theta_{a\bar{b}}\right) = \frac{1}{2R_1^2}\left[m_1-2i+\left(\phi_{i\alpha}^{(1)}\right)^\dagger\phi_{i\alpha}^{(1)}-\phi_{i+1\alpha}^{(1)}\left(\phi_{i+1\alpha}^{(1)}\right)^\dagger\right] + \frac{1}{2\,R_2^2}\left[m_2-2\alpha+\left(\phi_{i\alpha}^{(2)}\right)^\dagger\phi_{i\alpha}^{(2)}-\phi_{i\,\alpha+1}^{(2)}\left(\phi_{i\,\alpha+1}^{(2)}\right)^\dagger\right], \tag{4.19}$$

$$\left[X_{\bar{a}}^{i\alpha}\,,\,X_{\bar{b}}^{i\alpha}\right] = 0\,, \tag{4.20}$$

$$X_{\bar{a}}^{i\alpha}\,\phi_{i+1\,\alpha}^{(1)}-\phi_{i+1\,\alpha}^{(1)}\,X_{\bar{a}}^{i+1\,\alpha} = 0\,, \tag{4.21}$$

$$X_{\bar{a}}^{i\alpha}\,\phi_{i\,\alpha+1}^{(2)}-\phi_{i\,\alpha+1}^{(2)}\,X_{\bar{a}}^{i\,\alpha+1} = 0\,, \tag{4.22}$$

$$\phi_{i+1\,\alpha}^{(1)}\,\phi_{i+1\,\alpha+1}^{(2)}-\phi_{i\,\alpha+1}^{(2)}\,\phi_{i+1\,\alpha+1}^{(1)} = 0\,. \tag{4.23}$$

4.2 Examples

Before proceeding with a more general analysis, we will provide some illustration of the meaning of the quiver vortex equations (4.7)–(4.11) through special cases and limiting solutions.

Chain vortex equations. Consider a holomorphic chain (3.24) with $(m_1, m_2) = (m, 0)$. Its equations, obtainable from (4.7)–(4.11) by taking $\phi_{i\,\alpha+1}^{(2)} = 0$ in the ansatz for $\mathcal{A}$ and $\mathcal{F}$, read

$$g^{a\bar{b}}\,F_{a\bar{b}}^{i} = \frac{1}{2\,R^2}\left(m-2i+\phi_i^\dagger\,\phi_i-\phi_{i+1}\,\phi_{i+1}^\dagger\right), \qquad F_{\bar{a}\bar{b}}^{i} = 0\,, \tag{4.24}$$

$$\bar{\partial}_{\bar{a}}\phi_{i+1} + A_{\bar{a}}^{i}\,\phi_{i+1} - \phi_{i+1}\,A_{\bar{a}}^{i+1} = 0 \qquad \text{for} \quad i=0,1,\ldots,m\,, \tag{4.25}$$

where $\phi_i := \phi_{i\,0}^{(1)}$, $A^i := A^{i0}$, $F^i := F^{i0}$ and $R = R_1$. Noncommutative chain vortex configurations solving (4.24) and (4.25) on $M = \mathbb{R}_\theta^{2n}$ were constructed in [18].

Holomorphic triples. For $m = 1$ the holomorphic chain (3.24) reduces to a holomorphic triple $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ [12] described by the equations

$$g^{a\bar{b}}\,F_{a\bar{b}}^{0} = +\frac{1}{2\,R^2}\left(1-\phi\phi^\dagger\right), \qquad F_{\bar{a}\bar{b}}^{0} = 0\,, \tag{4.26}$$

$$g^{a\bar{b}}\,F_{a\bar{b}}^{1} = -\frac{1}{2\,R^2}\left(1-\phi^\dagger\phi\right), \qquad F_{\bar{a}\bar{b}}^{1} = 0\,, \tag{4.27}$$

$$\bar{\partial}_{\bar{a}}\phi + A_{\bar{a}}^{0}\,\phi - \phi\,A_{\bar{a}}^{1} = 0\,. \tag{4.28}$$

Solutions of (4.26)–(4.28) for $M = \mathbb{R}_\theta^{2n}$ and their D-brane interpretation were presented in [16, 17].

Four-dimensional case. For $\dim_{\mathbb{R}} M = 4$, $k_0 = k_1 = r$ and $\phi = \mathbf{1}_r$, we infer from (4.28) that $A^0 = A^1$, hence both (4.26) and (4.27) simplify to the self-dual Yang-Mills equations on M. In the case of $M = \mathbb{R}_\theta^4$ their solutions are noncommutative instantons (see e.g. [22, 23] and references therein). In string theory they are interpreted as states of noncommutative D-branes (see e.g. [24] and references therein). On the other hand, when $k_0 = k_1 = 1$ and ϕ is non-constant eqs. (4.26)–(4.28) reduce to the perturbed abelian

Seiberg-Witten monopole equations [25]. For $M = \mathbb{R}^4_\theta$ one encounters the noncommutative $\mathrm{U}_+(1)\times\mathrm{U}_-(1)$ Seiberg-Witten monopole equations studied in [26].

Vortices in two dimensions. For $\dim_{\mathbb{R}} M = 2$ and $k_0 = k_1 = 1$, the set (4.26)–(4.28) coincides with the standard vortex equations, whose solutions on $M = \mathbb{R}^2_\theta$ were considered e.g. in [27].

Quiver Toda equations. Let us investigate the equations (4.7)–(4.11) in the limit $R_1, R_2 \to \infty$ which decompactifies the spherical parts of our Kähler manifold $M\times\mathbb{C}P^1\times\mathbb{C}P^1$. With the redefinitions $\phi^{(\ell)}_{i\alpha} \to R_\ell\,\phi^{(\ell)}_{i\alpha}$ for $i = 0, 1, \dots, m_1$ and $\alpha = 0, 1, \dots, m_2$, the quiver vortex equations then descend to the *quiver Toda equations*

$$2g^{a\bar b}\,F^{i\alpha}_{a\bar b} \;=\; \big(\phi^{(1)}_{i\alpha}\big)^\dagger\,\phi^{(1)}_{i\alpha} \;-\; \phi^{(1)}_{i+1\,\alpha}\,\big(\phi^{(1)}_{i+1\,\alpha}\big)^\dagger + \big(\phi^{(2)}_{i\alpha}\big)^\dagger\,\phi^{(2)}_{i\alpha} \;-\; \phi^{(2)}_{i\,\alpha+1}\,\big(\phi^{(2)}_{i\,\alpha+1}\big)^\dagger\,, \tag{4.29}$$

$$F^{i\alpha}_{\bar a\bar b} = 0\,, \tag{4.30}$$

$$\partial_{\bar a}\phi^{(1)}_{i+1\,\alpha} + A^{i\alpha}_{\bar a}\,\phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha}\,A^{i+1\,\alpha}_{\bar a} = 0\,, \tag{4.31}$$

$$\partial_{\bar a}\phi^{(2)}_{i\,\alpha+1} + A^{i\alpha}_{\bar a}\,\phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,A^{i\,\alpha+1}_{\bar a} = 0\,, \tag{4.32}$$

$$\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1} = 0\,. \tag{4.33}$$

In this limit the induced quiver gauge theory on M is independent of the additional spherical dimensions. In the case $\phi^{(2)}_{i\alpha} = 0 \;\;\forall i,\alpha$ and $\phi_i := \phi^{(1)}_{i\,0}$ we arrive at

$$2g^{a\bar b}\,F^i_{a\bar b} \;=\; \phi_i^\dagger\,\phi_i - \phi_{i+1}\,\phi_{i+1}^\dagger\,, \qquad F^i_{\bar a\bar b} \;=\; 0\,, \qquad \bar\partial_{\bar a}\phi_{i+1} \;+\; A^i_{\bar a}\,\phi_{i+1} \;-\; \phi_{i+1}\,A^{i+1}_{\bar a} \;=\; 0\,, \tag{4.34}$$

which may be called the *holomorphic chain Toda equations* on the Kähler manifold M.

Symmetric instantons on $\mathbb{C}\boldsymbol{P^1}\times\mathbb{C}\boldsymbol{P^1}$. A somewhat opposite limit to the decompactification limit above comes from choosing the vacuum solution for generic monopole charges (m_1, m_2) on $M\times\mathbb{C}P^1\times\mathbb{C}P^1$. Let us set $A^{i\alpha} = 0$ in (2.23), $\phi^{(1)}_{i+1\,\alpha}$ and $\phi^{(2)}_{i\,\alpha+1}$ to constant matrices in (2.24)–(2.27), and $F^{i\alpha} = 0$ in (2.30). Then the field strength components (2.31)–(2.34) are identically zero, but (2.35)–(2.38) are generically non-vanishing. The components (2.41)–(2.43) vanish, while (2.44)–(2.47) are non-vanishing and give the components of the gauge fields on $\mathbb{C}P^1\times\mathbb{C}P^1$. The BPS equations (4.8)–(4.10) are identically satisfied in this case, while eqs. (4.7) and (4.11) should be solved with constant matrices $\phi^{(\ell)}_{i\alpha}$. The simplest choice is square matrices with $(m_1, m_2) = (1, 1)$. The BPS equations (4.7) and (4.11) are respectively equivalent in this case to the equations

$$\begin{aligned} \mathcal{F}^{i\alpha,i\alpha}_{y_1\bar y_1} + \mathcal{F}^{i\alpha,i\alpha}_{y_2\bar y_2} &= 0\,, \\ \mathcal{F}^{i+1\,\alpha+1,i\alpha}_{y_1 y_2} = 0 \;&=\; \mathcal{F}^{i\alpha,i+1\,\alpha+1}_{\bar y_1\bar y_2}\,. \end{aligned} \tag{4.35}$$

Furthermore, $\mathcal{F}^{i\,\alpha+1,i+1\,\alpha}_{y_1\bar y_2}$ is given by (2.47). The equations (4.35) give $\mathrm{SU}(2)\times\mathrm{SU}(2)$-equivariant solutions of the self-dual Yang-Mills equations on $\mathbb{C}P^1\times\mathbb{C}P^1$ which are vacuum BPS solutions of the original DUY equations. These solutions have non-zero energy, and the entire structure of these non-abelian instantons on $\mathbb{C}P^1\times\mathbb{C}P^1$ is reduced to equations for finite-dimensional matrices from our equivariant fields.

4.3 Finite energy solutions

Let us fix monopole charges $m_1, m_2 > 0$ and an arbitrary integer $0 < r \leq k$. Consider the ansatz

$$X_a^{i\alpha} = \theta_{a\bar{b}}\, T_{N_{i\alpha}}\, \bar{z}^{\bar{b}}\, T_{N_{i\alpha}}^\dagger \qquad \text{and} \qquad X_{\bar{a}}^{i\alpha} = \theta_{\bar{a}b}\, T_{N_{i\alpha}}\, z^b\, T_{N_{i\alpha}}^\dagger\,, \tag{4.36}$$

$$\phi_{i+1\,\alpha}^{(1)} = \lambda_{i+1\,\alpha}^{(1)}\, T_{N_{i\alpha}}\, T_{N_{i+1\,\alpha}}^\dagger \qquad \text{and} \qquad \big(\phi_{i+1\,\alpha}^{(1)}\big)^\dagger = \bar{\lambda}_{i+1\,\alpha}^{(1)}\, T_{N_{i+1\,\alpha}}\, T_{N_{i\alpha}}^\dagger\,, \tag{4.37}$$

$$\phi_{i\,\alpha+1}^{(2)} = \lambda_{i\,\alpha+1}^{(2)}\, T_{N_{i\alpha}}\, T_{N_{i\,\alpha+1}}^\dagger \qquad \text{and} \qquad \big(\phi_{i\,\alpha+1}^{(2)}\big)^\dagger = \bar{\lambda}_{i\,\alpha+1}^{(2)}\, T_{N_{i\,\alpha+1}}\, T_{N_{i\alpha}}^\dagger\,, \tag{4.38}$$

where $\lambda_{i\alpha}^{(1)}, \lambda_{i\alpha}^{(2)} \in \mathbb{C}$ are some constants with $\lambda_{0\alpha}^{(1)} = 0 = \lambda_{m_1+1\,\alpha}^{(1)}$ and $\lambda_{i0}^{(2)} = 0 = \lambda_{i\,m_2+1}^{(2)}$ for $i = 0, 1, \ldots, m_1$, $\alpha = 0, 1, \ldots, m_2$. Denoting by $\mathcal{H}$ the n-oscillator Fock space as before, the Toeplitz operators

$$T_{N_{i\alpha}} \;:\quad \mathbb{C}^r \otimes \mathcal{H} \;\longrightarrow\; \underline{V}_{k_{i\alpha}} \otimes \mathcal{H} \tag{4.39}$$

are partial isometries described by *rectangular* $k_{i\alpha} \times r$ matrices (with values in End $\mathcal{H}$) possessing the properties

$$T_{N_{i\alpha}}^\dagger\, T_{N_{i\alpha}} = \mathbf{1}_r \qquad \text{while} \qquad T_{N_{i\alpha}}\, T_{N_{i\alpha}}^\dagger = \mathbf{1}_{k_{i\alpha}} - P_{N_{i\alpha}}\,, \tag{4.40}$$

where $P_{N_{i\alpha}}$ is a hermitean projector of finite rank $N_{i\alpha}$ on the Fock space $\underline{V}_{k_{i\alpha}} \otimes \mathcal{H}$ so that

$$P_{N_{i\alpha}}^2 = P_{N_{i\alpha}} = P_{N_{i\alpha}}^\dagger \qquad \text{and} \qquad \mathrm{Tr}_{\underline{V}_{k_{i\alpha}} \otimes \mathcal{H}}\, P_{N_{i\alpha}} = N_{i\alpha}\,. \tag{4.41}$$

It follows that

$$\ker T_{N_{i\alpha}} = \{0\} \qquad \text{but} \qquad \ker T_{N_{i\alpha}}^\dagger = \mathrm{im}\, P_{N_{i\alpha}} \cong \mathbb{C}^{N_{i\alpha}}\,. \tag{4.42}$$

For the ansatz (4.36)–(4.38) the equations (4.20)–(4.23) are satisfied along with the non-holomorphic relations

$$\big(\phi_{i\alpha}^{(2)}\big)^\dagger\, \phi_{i+1\,\alpha-1}^{(1)} - \phi_{i+1\,\alpha}^{(1)}\, \big(\phi_{i+1\,\alpha}^{(2)}\big)^\dagger = 0\,, \tag{4.43}$$

or equivalently in terms of graded connections one has the commutativity condition

$$\Big[\boldsymbol{\phi}_{(m_1,\,m_2)}^{(1)}\,,\, \big(\boldsymbol{\phi}_{(m_1,\,m_2)}^{(2)}\big)^\dagger\Big] = 0\,. \tag{4.44}$$

The non-vanishing gauge field strength components are given by

$$F_{a\bar{b}}^{i\alpha} = \theta_{a\bar{b}}\, P_{N_{i\alpha}} = \frac{1}{2\,\theta^a}\, \delta_{a\bar{b}}\, P_{N_{i\alpha}}\,. \tag{4.45}$$

It follows that our ansatz determines a *finite-dimensional* representation of the quiver with relations $(\mathsf{Q}_{(m_1,m_2)}\,,\, \mathsf{R}_{(m_1,m_2)})$. The projectors $P_{N_{i\alpha}}$ give representations of the trivial path idempotents $e_{i\alpha}$ and project the infinite-dimensional Fock module $\underline{\mathcal{V}} \otimes \mathcal{H}$ over the path algebra $\mathsf{A}_{(m_1,m_2)}$, given by the noncommutative quiver bundle, onto finite-dimensional vector spaces $P_{N_{i\alpha}} \cdot (\underline{\mathcal{V}} \otimes \mathcal{H}) = \ker T_{N_{i\alpha}}^\dagger$. This module will be denoted as

$$\underline{\mathcal{I}} := \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \ker T_{N_{i\alpha}}^\dagger \tag{4.46}$$

with dimension vector

$$\vec{N} \;:=\; \vec{k}_{\underline{\mathcal{I}}} \;=\; \big(N_{i\alpha}\big)_{\alpha=0,1,\dots,m_2}^{i=0,1,\dots,m_1} \,. \tag{4.47}$$

These dimensions correspond to the degrees of the corresponding noncommutative subbundles determined by (4.45).

The noncommutative Yang-Mills action for the ansatz (4.36)–(4.38) can be evaluated by using (2.50), (2.58), (4.23), (4.40), (4.43) and (4.45) to get

$$\begin{aligned} S_{\rm YM} = -\pi\,R_1^2\,R_2^2\,{\rm Pf}(2\pi\,\theta)\,\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}\,{\rm Tr}_{\underline{V}_{k_{i\alpha}}\otimes\mathcal{H}}\Big\{\;{\rm tr}_{2n\times 2n}\big(\theta^{-2}\big)\;\;P_{N_{i\alpha}} \\ -\frac{1}{2\,R_1^4}\,\Big[(m_1-2i)\,\mathbf{1}_{k_{i\alpha}}+\Big(\big|\lambda_{i\alpha}^{(1)}\big|^2-\big|\lambda_{i+1\,\alpha}^{(1)}\big|^2\Big)\,\big(\mathbf{1}_{k_{i\alpha}}-P_{N_{i\alpha}}\big)\Big]^2 \\ -\frac{1}{2\,R_2^4}\,\Big[(m_2-2\alpha)\,\mathbf{1}_{k_{i\alpha}}+\Big(\big|\lambda_{i\alpha}^{(2)}\big|^2-\big|\lambda_{i\,\alpha+1}^{(2)}\big|^2\Big)\,\big(\mathbf{1}_{k_{i\alpha}}-P_{N_{i\alpha}}\big)\Big]^2\;\Big\}\,. \end{aligned} \tag{4.48}$$

Requiring that $S_{\rm YM}<\infty$ yields a pair of equations determining the moduli of the complex coefficients $\lambda_{i\alpha}^{(1)}$ and $\lambda_{i\alpha}^{(2)}$ respectively. Up to a phase they are thus uniquely fixed, by demanding that the ansatz (4.36)–(4.38) be a finite energy field configuration, as

$$\big|\lambda_{i\alpha}^{(1)}\big|^2 \;=\; i\,(m_1-i+1) \qquad\text{and}\qquad \big|\lambda_{i\alpha}^{(2)}\big|^2 \;=\; \alpha\,(m_2-\alpha+1)\,. \tag{4.49}$$

The corresponding finite action (4.48) then reads

$$\begin{aligned} S_{\rm YM} = \pi\,R_1^2\,R_2^2\,{\rm Pf}(2\pi\,\theta)\,\sum_{i=0}^{\lfloor\frac{m_1}{2}\rfloor}\,\sum_{\alpha=0}^{\lfloor\frac{m_2}{2}\rfloor}\,\big(N_{i\alpha}+N_{m_1-i\,m_2-\alpha}+N_{m_1-i\,\alpha}+N_{i\,m_2-\alpha}\big) \\ \times\,\Big[\;\frac{(m_1-2i)^2}{2\,R_1^4}+\frac{(m_2-2\alpha)^2}{2\,R_2^4}-{\rm tr}_{2n\times 2n}\big(\theta^{-2}\big)\;\Big]\,, \end{aligned} \tag{4.50}$$

where we have split the sum over nodes of the quiver $\mathsf{Q}_{(m_1,m_2)}$ into contributions from Dirac monopoles and antimonopoles which each have the same Yang-Mills energies on the spheres $\mathbb{C}P^1_{(1)}$ and $\mathbb{C}P^1_{(2)}$. This splitting will be the crux later on for the physical interpretation of our instanton solutions.

Finally, let us check that the Yang-Mills equations on $\mathbb{R}^{2n}_\theta\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ are indeed satisfied by our choice of ansatz. We have

$$\mathcal{A}_a-\theta_{a\bar b}\,\bar z^{\bar b} \;=\; \sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}\,X_a^{i\alpha}\otimes\Pi_{i\alpha} \;=\; \theta_{a\bar b}\,\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}\,T_{N_{i\alpha}}\,\bar z^{\bar b}\,T_{N_{i\alpha}}^\dagger\otimes\Pi_{i\alpha}\,, \tag{4.51}$$

$$\mathcal{A}_{\bar a}-\theta_{\bar a b}\,z^b \;=\; \sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}\,X_{\bar a}^{i\alpha}\otimes\Pi_{i\alpha} \;=\; \theta_{\bar a b}\,\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}\,T_{N_{i\alpha}}\,z^b\,T_{N_{i\alpha}}^\dagger\otimes\Pi_{i\alpha}\,, \tag{4.52}$$

while $\mathcal{A}_{y_1}$, $\mathcal{A}_{y_2}$, $\mathcal{A}_{\bar y_1}$ and $\mathcal{A}_{\bar y_2}$ are given by (3.20)–(3.23). For our ansatz the field strength

tensor has components

$$\mathcal{F}_{a\bar{b}} = \theta_{a\bar{b}} \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} P_{N_{i\alpha}} \otimes \Pi_{i\alpha} \,, \tag{4.53}$$

$$\mathcal{F}_{y_1\bar{y}_1} = \frac{1}{(1+y_1\bar{y}_1)^2} \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} (m_1 - 2i)\; P_{N_{i\alpha}} \otimes \Pi_{i\alpha} \,, \tag{4.54}$$

$$\mathcal{F}_{y_2\bar{y}_2} = \frac{1}{(1+y_2\bar{y}_2)^2} \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} (m_2 - 2\alpha)\; P_{N_{i\alpha}} \otimes \Pi_{i\alpha} \,, \tag{4.55}$$

where we have imposed the finite energy conditions (4.49). One can now easily check in the same way as in [18] that the Yang-Mills equations (4.1) are satisfied.

4.4 BPS solutions

The configurations described above are generically non-BPS solutions of the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times \mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$. Let us now describe the structure of the BPS states. Substituting (4.37), (4.38) and (4.45) into the remaining DUY equations (4.19) and using the finite energy constraints (4.49), one finds the BPS conditions

$$\sum_{a=1}^{n} \frac{1}{\theta^a} = \frac{m_1 - 2i}{2\,R_1^2} + \frac{m_2 - 2\alpha}{2\,R_2^2} \tag{4.56}$$

for all i, α with $N_{i\alpha} > 0$. Generically, these conditions are incompatible with one another unless only one of the degrees, say N_{00} for definiteness, is non-zero. Then the solution (4.36)–(4.38) is truncated by setting $T_{N_{i\alpha}} = \mathbf{1}_r$ for all $(i,\alpha) \neq (0,0)$ which correspond to vacuum gauge potentials $A^{i\alpha} = 0$ with trivial bundle maps $\phi^{(\ell)}_{i\alpha}$ acting as multiplication by the complex numbers $\lambda^{(\ell)}_{i\alpha}$ satisfying (4.49). The BPS solutions are also restricted to the special class of quiver representations (2.6) having dimension vectors $\vec{k}$ with $k_{i\alpha} = r \;\; \forall (i,\alpha) \neq (0,0)$ and $k_{00} + m_1\, m_2\, r = k$. As we will see in section 4.5, these quiver representations are essentially generic and hence BPS solutions always exist. The corresponding BPS energy (4.50) is proportional to the degree N_{00} and corresponds to the topological invariants displayed in (4.14), with the remaining terms vanishing due to the non-holomorphic relations (4.43).

Notice that there are special points in the quiver vortex moduli space where the generic BPS gauge symmetry $\mathrm{U}(k_{00}) \times \mathrm{U}(r)^{m_1\, m_2}$ is enhanced. For example, if $R_1 = R_2$ and p is any fixed integer with $0 \leq p \leq \min(m_1, m_2)$, then a BPS solution with $N_{i\,p-i} > 0$ for $i = 0, 1, \ldots, p$ is possible. This solution corresponds to a holomorphic chain along the diagonal vertices (i,α) of the quiver $\mathsf{Q}_{(m_1,m_2)}$ with $i + \alpha = p$. The corresponding BPS energies depend on p and are minimized precisely at $p = 0$.

The BPS solution having $N_{i\alpha} > 0$ may be characterized in quiver gauge theory as $N_{i\alpha}$ copies of the simple Schur representation $\underline{\mathcal{L}}_{i\alpha}$ for each $i = 0, 1, \ldots, m_1$, $\alpha = 0, 1, \ldots, m_2$. This is the $\mathsf{Q}_{(m_1,m_2)}$-module given by a one-dimensional vector space at vertex $(m_1 - 2i, m_2 - 2\alpha)$ with all maps equal to 0, i.e. the $\mathsf{A}_{(m_1,m_2)}$-module with $(\underline{\mathcal{L}}_{i\alpha})_{j\beta} = \delta_{ij}\, \delta_{\alpha\beta}\, \mathbb{C}$

and dimension vector $(\vec{k}_{\underline{\mathcal{L}}_{i\alpha}})_{j\beta} = \delta_{ij}\,\delta_{\alpha\beta}$. The generic non-BPS configurations give modules $\underline{\mathcal{I}}$ which are extensions of the BPS modules $(\underline{\mathcal{L}}_{i\alpha})^{\oplus N_{i\alpha}}$ [18] describing noncommutative quiver vortex configurations.

4.5 Instanton moduli space

We will now describe the moduli space of the generic (non-BPS) solutions that we have obtained. The equations of motion are fixed first of all by the positive integers n and k. The condition of G-equivariance then specifies a quiver representation (2.6) with dimension vector $\vec{k}$. The Yang-Mills action (4.50) is independent of $\vec{k}$, and later on we will find that in fact no physical quantities depend on the particular choice of quiver representation. As we now proceed to demonstrate, this independence is due to the triviality of the moduli space of $\mathsf{Q}_{(m_1,m_2)}$-modules.

Let us fix a dimension vector $\vec{k}$. Then with the identifications $\underline{V}_{k_{i\alpha}} \cong \mathbb{C}^{k_{i\alpha}}$ we can regard the module (2.6) as an element in the space of quiver representations *into* $\underline{\mathcal{V}}$ given by

$$\mathrm{Rep}\big(\mathsf{Q}_{(m_1,m_2)}\,,\vec{k}\,\big) := \bigoplus_{i=0}^{m_1}\,\bigoplus_{\alpha=0}^{m_2}\,\Big(\mathrm{Hom}\big(\mathbb{C}^{k_{i+1\,\alpha}}\,,\,\mathbb{C}^{k_{i\alpha}}\big)\oplus\mathrm{Hom}\big(\mathbb{C}^{k_{i\,\alpha+1}}\,,\,\mathbb{C}^{k_{i\alpha}}\big)\Big) \tag{4.57}$$

with $k_{m_1+1\,\alpha} := 0 =: k_{i\,m_2+1}$. This is the space of representations with fixed dimension vector $\vec{k}$. The set of representations of $\mathsf{Q}_{(m_1,m_2)}$ into $\underline{\mathcal{V}}$ satisfying the relations $\mathsf{R}_{(m_1,m_2)}$ is an affine variety inside the space (4.57).

The gauge group of the corresponding quiver gauge theory is given by (2.7). As in section 2.2, it is useful to work instead with the complexified gauge group

$$\mathsf{G}(\vec{k}\,) = \prod_{i=0}^{m_1}\,\prod_{\alpha=0}^{m_2}\,\mathrm{GL}(k_{i\alpha},\mathbb{C})\ . \tag{4.58}$$

Suppose that $\underline{\mathcal{V}}$, $\underline{\mathcal{V}}' \in \mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ and $\underline{f}\,:\underline{\mathcal{V}}\to\underline{\mathcal{V}}'$ is an isomorphism of quiver representations. Then $\underline{f}$ can be naturally regarded as an element of the gauge group (4.58). Conversely, any element $\underline{f} = \{f_{i\alpha}\in\mathrm{GL}(k_{i\alpha},\mathbb{C})\}_{0\le i\le m_1,0\le\alpha\le m_2}\in\mathsf{G}(\vec{k}\,)$ acts on $\underline{\mathcal{V}}\in\mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ in the same fashion. It follows that the gauge group $\mathsf{G}(\vec{k}\,)$ acts on $\mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ and two quiver representations are isomorphic if and only if they lie in the same orbit of $\mathsf{G}(\vec{k}\,)$. Thus there is a one-to-one correspondence between $\mathsf{G}(\vec{k}\,)$-orbits in $\mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ and isomorphism classes of $\mathsf{Q}_{(m_1,m_2)}$-modules with dimension vector $\vec{k}$.

This set defines the moduli space $\mathcal{M}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ of quiver representations. It has virtual dimension [28]

$$\begin{aligned}\dim\Big[\mathcal{M}\big(\mathsf{Q}_{(m_1,m_2)}\,,\,\vec{k}\,\big)\Big]^{\mathrm{vir}} &= 1+\dim\,\mathrm{Rep}\big(\mathsf{Q}_{(m_1,m_2)}\,,\vec{k}\,\big)-\dim\,\mathsf{G}\big(\vec{k}\,\big)\\ &= 1-\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}\,k_{i\alpha}\,\big(k_{i\alpha}-k_{i+1\,\alpha}-k_{i\,\alpha+1}\big)\ .\end{aligned} \tag{4.59}$$

Restricting to representations which satisfy the relations $\mathsf{R}_{(m_1,m_2)}$ lowers (4.59) by $\sum_{i,\alpha} \times k_{i\alpha}k_{i+1\,\alpha+1}$. Representations with moduli space dimension greater than the virtual dimension can arise due to additional unbroken gauge symmetry, as described in section 4.4. Schur representations, describing generic BPS states, are those modules for which the stable dimension equals the virtual dimension. Rigid representations carry no moduli and have vanishing virtual dimension. As we now show, it is these latter $\mathsf{Q}_{(m_1,m_2)}$-modules that parametrize our noncommutative quiver vortices.

The scalar subgroup $\mathbb{C}^\times \subset \mathsf{G}(\vec{k}\,)$ acts trivially on $\mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$, and we are left with a free action of the projective gauge group $\mathsf{PG}(\vec{k}\,) := \mathsf{G}(\vec{k}\,)/\mathbb{C}^\times$. Since $\mathsf{PG}(\vec{k}\,)$ is not compact, we must use geometric invariant theory to obtain a quotient which is well-defined as a projective variety [29]. The representation space $X = \mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ is an affine variety. Let $\mathbb{C}[X]$ denote the ring of polynomial functions on X. The $\mathsf{PG}(\vec{k}\,)$-action on X induces a $\mathsf{PG}(\vec{k}\,)$-action on $\mathbb{C}[X]$ in the usual way by pull-back. Let $\mathbb{C}[X]^{\mathsf{PG}(\vec{k}\,)} \subset \mathbb{C}[X]$ be the subalgebra of $\mathsf{PG}(\vec{k}\,)$-invariant polynomials. Since the gauge group (4.58) is reductive, the graded ring $\mathbb{C}[X]^{\mathsf{PG}(\vec{k}\,)}$ is finitely generated and by the Gel'fand-Naimark theorem it can be regarded as the polynomial ring of a complex projective affine variety $X /\!\!/ \mathsf{PG}(\vec{k}\,)$. This defines the desired moduli space

$$\mathcal{M}\big(\mathsf{Q}_{(m_1,m_2)}\,,\,\vec{k}\,\big) \;:=\; \mathrm{Rep}\big(\mathsf{Q}_{(m_1,m_2)}\,,\,\vec{k}\,\big) /\!\!/ \mathsf{PG}(\vec{k}\,) \;=\; \mathrm{Proj}\,\mathbb{C}\big[\mathrm{Rep}\big(\mathsf{Q}_{(m_1,m_2)}\,,\,\vec{k}\,\big)\big]^{\mathsf{PG}(\vec{k}\,)}\,. \tag{4.60}$$

Now since the quiver $\mathsf{Q}_{(m_1,m_2)}$ has no oriented cycles, we may lexicographically order its vertex set as $\mathsf{Q}^{(0)}_{(m_1,m_2)} = \{1,2,\dots,(m_1+1)\,(m_2+1)\}$ and assume that the integer label of the tail node of each arrow is smaller than that of the head node. For $\zeta \in \mathbb{C}^\times$ we define $\underline{f}_\zeta \in \mathsf{G}(\vec{k}\,)$ by $(\,\underline{f}_\zeta)_i = \zeta^i\;\mathbf{1}_{k_i} \in \mathrm{GL}(k_i,\mathbb{C})$ for each $i \in \mathsf{Q}^{(0)}_{(m_1,m_2)}$. Then by considering the action of $\underline{f}_\zeta$ on $X = \mathrm{Rep}(\mathsf{Q}_{(m_1,m_2)},\vec{k}\,)$ and on $\mathbb{C}[X]^{\mathsf{PG}(\vec{k}\,)}$, one easily deduces that $\mathbb{C}[X]^{\mathsf{PG}(\vec{k}\,)} \cong \mathbb{C}$. This means that the moduli space (4.60) is trivial,

$$\mathcal{M}\big(\mathsf{Q}_{(m_1,m_2)}\,,\,\vec{k}\,\big) = \mathrm{point}\,, \tag{4.61}$$

and all quiver representations are gauge equivalent.

Thus the only moduli of our solutions arise from the moduli space of noncommutative solitons [30]. They are parametrized by the pair of monopole charges (m_1,m_2) and by the dimension vector $\vec{N}$ of the quiver representation (4.46). The above argument again shows that there are no extra moduli associated with the $\mathsf{Q}_{(m_1,m_2)}$-modules $\underline{\mathcal{I}}$. For each i,α we let $b_{l_{i\alpha}} = (b^a_{l_{i\alpha}})$, $l_{i\alpha} = 1,\dots,N_{i\alpha}$ be the holomorphic components of fixed points in $\mathbb{C}^n$, and let $|b_{l_{i\alpha}}\rangle$ be the corresponding coherent states in the n-oscillator Fock space $\mathcal{H}$. For the projector $P_{N_{i\alpha}}$ in the solution of section 4.3 we may take the orthogonal projection of $\mathcal{H}$ onto the linear span $\bigoplus_{l_{i\alpha}=1}^{N_{i\alpha}} \mathbb{C}|b_{l_{i\alpha}}\rangle$. Modulo the standard action of the noncommutative gauge group $\mathrm{U}(\mathcal{H}) \cong \mathrm{U}(\infty)$, the moduli space of these projectors can be described as an ideal $\mathcal{I}$ of the ring of polynomials $\mathbb{C}[\bar{z}^1,\dots,\bar{z}^n]$ in the noncommutative coordinates acting on the vacuum state $|0,\dots,0\rangle$. The zero set of $\mathcal{I}$ gives the locations of the instantons in $\mathbb{C}^n$ and the codimension of $\mathcal{I}$ in $\mathbb{C}[\bar{z}^1,\dots,\bar{z}^n]$ is the number $N_{i\alpha}$ of instantons. The moduli space of partial isometries $T_{N_{i\alpha}}$ thereby coincides with the Hilbert scheme $\mathrm{Hilb}^{N_{i\alpha}}(\mathbb{C}^n)$

of $N_{i\alpha}$ points in $\mathbb{C}^n$ [30], and thus the total moduli space of the solutions constructed in section 4.3 is

$$\mathcal{M}^n_{(m_1,m_2)}(\vec{N}\,) = \prod_{i=0}^{m_1}\,\prod_{\alpha=0}^{m_2}\,\mathrm{Hilb}^{N_{i\alpha}}(\mathbb{C}^n)\ . \tag{4.62}$$

The quiver representation (4.46) thereby specifies the supports of the noncommutative quiver vortices in $\mathbb{R}^{2n}$. Explicit forms for the Toeplitz operators $T_{N_{i\alpha}}$ corresponding to specific points in (4.62) may be constructed exactly as in [18] by using the noncommutative ABS construction. We will return to this point in the next section.

5. D-brane realizations

In this final section we will elucidate the physical interpretation of our solutions as particular configurations of branes and antibranes in Type IIA superstring theory. We will first compute, in the original gauge theory on $\mathbb{R}^{2n}_\theta\times\mathbb{C}P^1\times\mathbb{C}P^1$, the topological charges of the multi-instanton solutions constructed in section 4.3. This will make clear the D-brane interpretation which we describe in detail. We then present two independent checks of the proposed identification. Firstly, we work out the K-theory charges associated to the noncommutative quiver vortices. Secondly, we compute the topological charge in the quiver gauge theory arising after dimensional reduction to $\mathbb{R}^{2n}_\theta$. While formally similar to the construction of [18] in the case of holomorphic chains, the new feature of the higher rank quiver is that all of these computations of D-brane charges agree *only* when one imposes the appropriate relations derived earlier. The ensuing calculations thereby also provide a nice physical realization of the quiver with relations $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$. Details of the homological algebra techniques used in this section may be found in [21, 31].

5.1 Topological charges

Let us compute the topological charge of the configurations (4.36)–(4.41). The non-vanishing components of the field strength tensor along $\mathbb{R}^{2n}_\theta$ are given by

$$\mathcal{F}_{2a-1\,2a} \;=\; 2\,\mathrm{i}\,\mathcal{F}_{a\bar{a}} \;=\; -\frac{\mathrm{i}}{\theta^a}\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}P_{N_{i\alpha}}\otimes\Pi_{i\alpha}\,, \tag{5.1}$$

while the non-vanishing spherical components can be written in terms of angular coordinates on $S^2_{(1)}\times S^2_{(2)}$ as

$$\mathcal{F}_{\vartheta_1\varphi_1} = -\,\mathrm{i}\,\frac{\sin\vartheta_1}{2}\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}(m_1-2i)\;P_{N_{i\alpha}}\otimes\Pi_{i\alpha}\,, \tag{5.2}$$

$$\mathcal{F}_{\vartheta_2\varphi_2} = -\,\mathrm{i}\,\frac{\sin\vartheta_2}{2}\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}(m_2-2\alpha)\;P_{N_{i\alpha}}\otimes\Pi_{i\alpha}\ . \tag{5.3}$$

This gives

$$\begin{aligned}
&\mathcal{F}_{12}\,\mathcal{F}_{34}\cdots\mathcal{F}_{2n-1\,2n}\,\mathcal{F}_{\vartheta_1\varphi_1}\,\mathcal{F}_{\vartheta_2\varphi_2}\\
&\quad=\ (-\mathrm{i}\,)^n\,\frac{\sin\vartheta_1\,\sin\vartheta_2}{4\,\mathrm{Pf}(\theta)}\,\Big(\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}P_{N_{i\alpha}}\otimes\Pi_{i\alpha}\Big)^n\\
&\qquad\times\Big(\sum_{j_1=0}^{m_1}\,\sum_{\gamma_1=0}^{m_2}(m_1-2j_1)\;P_{N_{j_1\gamma_1}}\otimes\Pi_{j_1\gamma_1}\Big)\Big(\sum_{j_2=0}^{m_1}\,\sum_{\gamma_2=0}^{m_2}(m_2-2j_2)\;P_{N_{j_2\gamma_2}}\otimes\Pi_{j_2\gamma_2}\Big)\\
&\quad=\ (-\mathrm{i}\,)^n\,\frac{\sin\vartheta_1\,\sin\vartheta_2}{4\,\mathrm{Pf}(\theta)}\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}(m_1-2i)\,(m_2-2\alpha)\;P_{N_{i\alpha}}\otimes\Pi_{i\alpha}\ .
\end{aligned}\tag{5.4}$$

The instanton charge is then given by the $(n+2)$-th Chern number

$$Q:=\frac{1}{(n+2)!}\,\Big(\frac{\mathrm{i}}{2\pi}\Big)^{n+2}\,\mathrm{Pf}(2\pi\,\theta)\int_{S^2_{(1)}\times S^2_{(2)}}\mathrm{Tr}_{\underline{\mathcal{V}}\otimes\mathcal{H}}\;\underbrace{\mathcal{F}\wedge\cdots\wedge\mathcal{F}}_{n+2}\ .\tag{5.5}$$

The calculation now proceeds exactly as in [18] and one finds

$$Q=\sum_{i=0}^{m_1}\,\sum_{\alpha=0}^{m_2}(m_1-2i)\,(m_2-2\alpha)\,N_{i\alpha}\ .\tag{5.6}$$

For the BPS configurations described in section 4.4 the energy functional (4.50) is proportional to the topological charge (5.6), as expected for a BPS instanton solution.

As we did in (4.50), let us rewrite (5.6) in the form

$$Q=\sum_{i=0}^{\lfloor\frac{m_1}{2}\rfloor}\,\sum_{\alpha=0}^{\lfloor\frac{m_2}{2}\rfloor}(m_1-2i)\,(m_2-2\alpha)\,\Big[\big(N_{i\,\alpha}+N_{m_1-i\,m_2-\alpha}\big)-\big(N_{m_1-i\,\alpha}+N_{i\,m_2-\alpha}\big)\Big]\ .\tag{5.7}$$

This formula suggests that one should regard the nodes of the quiver bundle (2.12) which live in the upper right and lower left quadrants as branes (with positive charges), and those in the upper left and lower right quadrants as antibranes (with negative charges). The branes and antibranes are realized as a quiver vortex configuration on $\mathbb{R}^{2n}_\theta$ of D0-branes in a system of $k=\sum_{i,\alpha}k_{i\alpha}$ D$(2n)$-branes. The twisting of the Chan-Paton bundles by the Dirac multi-monopole bundles over the $\mathbb{C}P^1$ factors is crucial in this construction. This system is equivalent to a configuration of spherical D2-branes, wrapping $\mathbb{C}P^1_{(\ell)}$ for $\ell=1,2$, inside a system of D$(2n+4)$-branes on $\mathbb{R}^{2n}_\theta\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$. The monopole flux through each $\mathbb{C}P^1$ factor stabilizes the D2-branes. After equivariant dimensional reduction, the D$(2n)$-branes which carry negative magnetic flux on their worldvolume have opposite orientation with respect to those which carry positive magnetic flux, and are thus antibranes. The bi-fundamental scalar fields $\phi^{(\ell)}_{i\alpha}$ correspond to massless open string excitations between nearest neighbour D-branes on the quiver $\mathsf{Q}_{(m_1,m_2)}$. The relations $\mathsf{R}_{(m_1,m_2)}$ of the quiver, given by (2.13), imply that there is a unique Higgs excitation marginally binding any given pair of D-branes. As will become apparent in section 5.3, only those brane-antibrane pairs whose total monopole charge vanishes are actually unstable and possess tachyonic

JHEP09(2006)054

excitations causing them to annihilate to the vacuum. Other pairs are stabilized by the non-trivial monopole bundles over the two $\mathbb{C}P^1$ factors which act as a source of flux stabilization. This interpretation is consistent with the form of the energy (4.50) of our solutions, and the stability of the brane configuration is consistent with the structure of BPS solutions found in section 4.4. In the remainder of this section we will justify and expand on these statements.

5.2 Symmetric spinors

The standard explicit realization of the basic partial isometry operators $T_{N_{i\alpha}}$ describing the noncommutative multi-instanton solutions is provided by a G-equivariant version of the (noncommutative) Atiyah-Bott-Shapiro (ABS) construction of tachyon field configurations [18], where $G = \mathrm{SU}(2)\times\mathrm{SU}(2)$. Let us now describe some general aspects of this construction. We begin with the equivariant excision theorem [32] which computes the G-equivariant K-theory of the space $M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$ through the isomorphism

$$\mathrm{K}_G\big(M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}\big) \;=\; \mathrm{K}_G(G\times_H M) \;=\; \mathrm{K}_H(M)\ . \tag{5.8}$$

Since the closed subgroup $H = \mathrm{U}(1)\times\mathrm{U}(1) \subset G$ acts trivially on M, from the Künneth theorem we arrive at

$$\mathrm{K}_G\big(M\times\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}\big) = \mathrm{K}(M)\otimes \mathrm{R}^{(1)}_{\mathrm{U}(1)}\otimes \mathrm{R}^{(2)}_{\mathrm{U}(1)}\,, \tag{5.9}$$

where $\mathrm{R}_{\mathrm{U}(1)}$ is the representation ring of the group $\mathrm{U}(1)$. Setting M = point in this isomorphism and using (2.14), we may describe this representation ring as the formal Laurent polynomial ring $\mathrm{R}_H = \mathrm{K}_G(\mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}) = \mathbb{Z}[\mathcal{L}_{(1)},\mathcal{L}^\vee_{(1)}]\otimes\mathbb{Z}[\mathcal{L}_{(2)},\mathcal{L}^\vee_{(2)}]$. Then (5.9) is just the generalization of the isomorphism described in section 2.2 to the case of virtual bundles.

In the case of main interest, $M = \mathbb{R}^{2n}$, we can make the above isomorphism very explicit. Let $\mathrm{R}_{\mathrm{Spin}_H(2n)}$ be the Grothendieck group of isomorphism classes of finite-dimensional $\mathbb{Z}_2$-graded $H\times\mathrm{C}\ell_{2n}$-modules, where $\mathrm{C}\ell_{2n} := \mathrm{C}\ell(\mathbb{R}^{2n})$ denotes the Clifford algebra of the vector space $\mathbb{R}^{2n}$ with the canonical inner product $\delta_{\mu\nu}$. Extending the standard ABS construction [33], we may then compute the H-equivariant K-theory $\mathrm{K}_H(\mathbb{R}^{2n})$ with H acting trivially on $\mathbb{R}^{2n}$ and commuting with the Clifford action. Any such $H\times\mathrm{C}\ell_{2n}$-module is a direct sum of products of an H-module and a spinor module, and hence

$$\mathrm{R}_{\mathrm{Spin}_H(2n)} = \mathrm{R}_{\mathrm{Spin}(2n)}\otimes \mathrm{R}^{(1)}_{\mathrm{U}(1)}\otimes \mathrm{R}^{(2)}_{\mathrm{U}(1)}\ . \tag{5.10}$$

The first factor can be treated by the standard ABS construction and yields the ordinary K-theory group $\mathrm{K}(\mathbb{R}^{2n})$. Therefore, our equivariant K-theory group reduces to

$$\mathrm{K}_H\big(\mathbb{R}^{2n}\big) = \mathrm{K}\big(\mathbb{R}^{2n}\big)\otimes \mathrm{R}^{(1)}_{\mathrm{U}(1)}\otimes \mathrm{R}^{(2)}_{\mathrm{U}(1)}\ . \tag{5.11}$$

In the present context of the equivariant ABS construction, this isomorphism may be described in terms of the isotopical decomposition of the spinor module

$$\underline{\Delta}_{2n} \;:=\; \underline{\Delta}\big(\mathbb{R}^{2n}\big) \;=\; \bigoplus_{i=0}^{m_1}\ \bigoplus_{\alpha=0}^{m_2} \Delta_{i\alpha}\otimes \underline{S}^{(1)}_{m_1-2i}\otimes \underline{S}^{(2)}_{m_2-2\alpha} \tag{5.12}$$

obtained by restricting $\underline{\Delta}_{2n}$ to representations of $\mathrm{U}(1)\times\mathrm{U}(1) \subset \mathrm{Spin}(2n) \subset \mathrm{C}\ell_{2n}$. Let $\iota : H \hookrightarrow G$ be the inclusion map. It induces a restriction map from representations of G to representations of H, and hence a homomorphism of representation rings

$$\iota^* \,:\, \mathrm{R}_G \;\longrightarrow\; \mathrm{R}_H \ . \tag{5.13}$$

The $\Delta_{i\alpha}$'s in (5.12) are then the corresponding multiplicity spaces

$$\Delta_{i\alpha} = \mathrm{Hom}_H\big(\iota^*\underline{\Delta}_{2n}\,,\,\underline{S}^{(1)}_{m_1-2i}\otimes\underline{S}^{(2)}_{m_2-2\alpha}\big)\ . \tag{5.14}$$

To compute the spaces (5.14) explicitly, consider the homomorphism of representation rings

$$\iota_* \,:\, \mathrm{R}_H \;\longrightarrow\; \mathrm{R}_G \tag{5.15}$$

induced by the induction map from representations of H to representations of G. On generators it is given by the space of sections

$$\iota_*\big(\,\underline{S}^{(1)}_{p_1}\otimes\underline{S}^{(2)}_{p_2}\big) = \Gamma\big(\mathcal{L}^{p_1}_{(1)}\otimes\mathcal{L}^{p_2}_{(2)}\big) \tag{5.16}$$

of the homogeneous line bundle $\mathcal{L}^{p_1}_{(1)}\otimes\mathcal{L}^{p_2}_{(2)} = G\times_H\big(\,\underline{S}^{(1)}_{p_1}\otimes\underline{S}^{(2)}_{p_2}\big)$ over the base space $G/H \cong \mathbb{C}P^1_{(1)}\times\mathbb{C}P^1_{(2)}$, with G-action induced by the standard action on the base. By Frobenius reciprocity we have $\dim\,\mathrm{Hom}_G(\underline{V}\,,\,\iota_*\underline{W}\,) = \dim\,\mathrm{Hom}_H(\iota^*\underline{V}\,,\,\underline{W}\,)$ for $\underline{V}$ a representation of G and $\underline{W}$ a representation of H. As a consequence we can identify the multiplicity spaces (5.14) as

$$\Delta_{i\alpha} = \mathrm{Hom}_G\Big(\,\underline{\Delta}_{2n}\,,\,\Gamma\big(\mathcal{L}^{m_1-2i}_{(1)}\otimes\mathcal{L}^{m_2-2\alpha}_{(2)}\big)\Big)\ . \tag{5.17}$$

We may now calculate the isotopical decomposition (5.12) by using (5.17) to construct the $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant dimensional reduction of spinors from $\mathbb{R}^{2n}\times\mathbb{C}P^1\times\mathbb{C}P^1$ to $\mathbb{R}^{2n}$. To this end, we introduce the twisted Dirac operator on $\mathbb{R}^{2n}\times\mathbb{C}P^1\times\mathbb{C}P^1$ using the graded connection formalism of section 3.4 to write the $\mathbb{Z}_{m_1+1}\times\mathbb{Z}_{m_2+1}$-graded Clifford connection

$$\begin{aligned}\hat{\not{D}} := \Gamma^{\hat{\mu}}\,D_{\hat{\mu}} \;=\; \gamma^\mu\,D_\mu\otimes\mathbf{1}_2\otimes\mathbf{1}_2 &+\big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\big)\,\gamma\otimes\gamma^{\bar{y}_1}\,\beta_{\bar{y}_1}\otimes\mathbf{1}_2 - \big(\boldsymbol{\phi}^{(1)}_{(m_1,\,m_2)}\big)^\dagger\,\gamma\otimes\gamma^{y_1}\,\beta_{y_1}\otimes\mathbf{1}_2\\ &+\big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\big)\,\gamma\otimes\mathbf{1}_2\otimes\gamma^{\bar{y}_2}\,\beta_{\bar{y}_2} - \big(\boldsymbol{\phi}^{(2)}_{(m_1,\,m_2)}\big)^\dagger\,\gamma\otimes\mathbf{1}_2\otimes\gamma^{y_2}\,\beta_{y_2}\\ &+\gamma\otimes\not{D}^{(1)}_{\mathbb{C}P^1}\otimes\mathbf{1}_2 + \gamma\otimes\mathbf{1}_2\otimes\not{D}^{(2)}_{\mathbb{C}P^1}\end{aligned} \tag{5.18}$$

where

$$\not{D}^{(\ell)}_{\mathbb{C}P^1} \;:=\; \gamma^{y_\ell}\Big(\partial_{y_\ell}+\omega_{y_\ell}+\big(\boldsymbol{a}^{(m_\ell)}\big)_{y_\ell}\Big) \;+\; \gamma^{\bar{y}_\ell}\Big(\partial_{\bar{y}_\ell}+\omega_{\bar{y}_\ell}+\big(\boldsymbol{a}^{(m_\ell)}\big)_{\bar{y}_\ell}\Big) \tag{5.19}$$

with $\ell = 1,2$, and $\omega_y, \omega_{\bar{y}}$ are the components of the Levi-Civita spin connection on the tangent bundle of $\mathbb{C}P^1$. The operator (5.18) acts on sections Ψ of the twisted spinor bundle

$$\not{\mathcal{S}} = \bigoplus_{i=0}^{m_1}\bigoplus_{\alpha=0}^{m_2}(E_{k_{i\alpha}}\otimes\underline{\Delta}_{2n})\otimes\begin{pmatrix}\mathcal{L}^{m_1-2i+1}_{(1)}\\ \mathcal{L}^{m_1-2i-1}_{(1)}\end{pmatrix}\otimes\begin{pmatrix}\mathcal{L}^{m_2-2\alpha+1}_{(2)}\\ \mathcal{L}^{m_2-2\alpha-1}_{(2)}\end{pmatrix} \tag{5.20}$$

over $\mathbb{R}^{2n}\times\mathbb{C}P^1\times\mathbb{C}P^1$, where $\mathcal{L}^{p+1}\oplus\mathcal{L}^{p-1}$ are the twisted spinor bundles of rank 2 over the sphere $\mathbb{C}P^1$. We are therefore interested in the product of the spinor module $\underline{\Delta}_{2n}\otimes$

$\underline{\Delta}(\mathbb{C}P^1) \otimes \underline{\Delta}(\mathbb{C}P^1)$ with the fundamental representation (2.6) of the gauge group $\mathrm{U}(k)$ broken as in (2.7).

The symmetric fermions on $\mathbb{R}^{2n}$ that we are interested in correspond to $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant spinors on $\mathbb{R}^{2n}\times\mathbb{C}P^1\times\mathbb{C}P^1$. They belong to the kernels $\ker(\not{D}^{(1)}_{\mathbb{C}P^1}) \otimes \ker(\not{D}^{(2)}_{\mathbb{C}P^1})$ of the two Dirac operators (5.19) on $\mathbb{C}P^1$. By using (3.15), (3.16) and (3.42) one can write chiral decompositions of the Dirac operators (5.19) acting on (5.20) in the form

$$\not{D}^{(1)}_{\mathbb{C}P^1} \;=\; \bigoplus_{i=0}^{m_1} \begin{pmatrix} 0 & \not{D}^{(1)\,+}_{m_1-2i} \\ \not{D}^{(1)\,-}_{m_1-2i} & 0 \end{pmatrix} \qquad \text{and} \qquad \not{D}^{(2)}_{\mathbb{C}P^1} \;=\; \bigoplus_{\alpha=0}^{m_2} \begin{pmatrix} 0 & \not{D}^{(2)\,+}_{m_2-2\alpha} \\ \not{D}^{(2)\,-}_{m_2-2\alpha} & 0 \end{pmatrix} , \tag{5.21}$$

where

$$\not{D}^{(1)\,+}_{m_1-2i} \;=\; -\frac{1}{R_1^2}\,\big[(R_1^2+y_1\bar{y}_1)\,\partial_{y_1} + \tfrac{1}{2}\,(m_1-2i+1)\,\bar{y}_1\big]\,, \tag{5.22}$$

$$\not{D}^{(1)\,-}_{m_1-2i} \;=\; \frac{1}{R_1^2}\,\big[(R_1^2+y_1\bar{y}_1)\,\partial_{\bar{y}_1} - \tfrac{1}{2}\,(m_1-2i+1)\,y_1\big] \tag{5.23}$$

and analogously for $\not{D}^{(2)\,\pm}_{m_2-2\alpha}$. The non-trivial kernels are naturally isomorphic to irreducible $\mathrm{SU}(2)$-modules [18] given by

$$\begin{aligned} \ker \not{D}^{(\ell)\,+}_{p} \;&=\; \{0\} \qquad \text{and} \qquad \ker \not{D}^{(\ell)\,-}_{p} \;=\; \underline{V}_{|p|} \quad \text{for} \quad p<0\,, \\ \ker \not{D}^{(\ell)\,+}_{p} \;&=\; \underline{V}_{p} \qquad \text{and} \qquad \ker \not{D}^{(\ell)\,-}_{p} \;=\; \{0\} \quad \text{for} \quad p>0\,, \end{aligned} \tag{5.24}$$

with $p = m_1 - 2i$ for $\ell = 1$ and $p = m_2 - 2\alpha$ for $\ell = 2$. Thus the chirality gradings are by the signs of the corresponding magnetic charges.

It follows that the $\mathrm{SU}(2)\times\mathrm{SU}(2)$-equivariant reduction of the twisted spinor representation of $\mathrm{C}\ell(\mathbb{R}^{2n}\times\mathbb{C}P^1\times\mathbb{C}P^1)$ decomposes as a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded bundle giving

$$\underline{\Delta}_{\underline{\mathcal{V}}}{}^{\mathrm{SU}(2)\times\mathrm{SU}(2)} = \underline{\Delta}_{2n} \otimes \big(\underline{\Delta}^{++}_{\underline{\mathcal{V}}} \;\oplus\; \underline{\Delta}^{+-}_{\underline{\mathcal{V}}} \;\oplus\; \underline{\Delta}^{-+}_{\underline{\mathcal{V}}} \;\oplus\; \underline{\Delta}^{--}_{\underline{\mathcal{V}}}\big)\,, \tag{5.25}$$

where

$$\begin{aligned} \underline{\Delta}^{++}_{\underline{\mathcal{V}}} \;&=\; \bigoplus_{i=0}^{m_1^-} \bigoplus_{\alpha=0}^{m_2^-} \underline{\Delta}_{i\alpha} \qquad \text{and} \qquad \underline{\Delta}^{+-}_{\underline{\mathcal{V}}} \;=\; \bigoplus_{i=0}^{m_1^-} \bigoplus_{\alpha=m_2^+}^{m_2} \underline{\Delta}_{i\alpha}\,, \\ \underline{\Delta}^{-+}_{\underline{\mathcal{V}}} \;&=\; \bigoplus_{i=m_1^+}^{m_1} \bigoplus_{\alpha=0}^{m_2^-} \underline{\Delta}_{i\alpha} \qquad \text{and} \qquad \underline{\Delta}^{--}_{\underline{\mathcal{V}}} \;=\; \bigoplus_{i=m_1^+}^{m_1} \bigoplus_{\alpha=m_2^+}^{m_2} \underline{\Delta}_{i\alpha} \end{aligned} \tag{5.26}$$

with

$$\underline{\Delta}_{i\alpha} \;=\; \underline{V}_{k_{i\alpha}} \otimes \underline{V}_{|m_1-2i|} \otimes \underline{V}_{|m_2-2\alpha|} \qquad \text{and} \qquad m_\ell^\pm \;=\; \big\lfloor \tfrac{m_\ell \pm 1}{2} \big\rfloor\,. \tag{5.27}$$

The reduction (5.25) is valid for $m_1\,m_2$ odd, which we henceforth assume for brevity. When $m_1\,m_2$ is even, one should also couple eigenspaces of spinor harmonics in the appropriate manner [18].

The chirality bi-grading in (5.25) is by the signs of the magnetic charges. The multiplicative $\mathbb{Z}_2$-grading induced by this $\mathbb{Z}_2\times\mathbb{Z}_2$-grading coincides with the grading into brane-antibrane pairs infered from (5.7). The corresponding actions of the two Clifford multiplications

$$\mu^{(1)}_{\underline{\mathcal{V}}} \,:\, \underline{\Delta}^{-\bullet}_{\underline{\mathcal{V}}} \;\longrightarrow\; \underline{\Delta}^{+\bullet}_{\underline{\mathcal{V}}} \qquad \text{and} \qquad \mu^{(2)}_{\underline{\mathcal{V}}} \,:\, \underline{\Delta}^{\bullet-}_{\underline{\mathcal{V}}} \;\longrightarrow\; \underline{\Delta}^{\bullet+}_{\underline{\mathcal{V}}} \tag{5.28}$$

are uniquely fixed on isotopical components in the same manner as in [18]. They give the tachyon fields which are maps between branes of equal and opposite charge.

The equivalence between D-brane charges on $M\times\mathbb{C}P^1\times\mathbb{C}P^1$ and on M asserted by the isomorphism (5.9) can now be understood heuristically through equivariant dimensional reduction as follows. The graded Clifford connection (5.18) defines a class $[\hat{\not{D}}]$ in the analytic K-homology group $\mathrm{K}^{\mathrm{a}}(M\times\mathbb{C}P^1\times\mathbb{C}P^1)$. Corresponding to $[\hat{\not{D}}]$, we may define a fermionic action functional on the space of sections Ψ of the bundle (5.20) by

$$S_{\mathrm{D}} := \int_{M\times\mathbb{C}P^1\times\mathbb{C}P^1} \mathrm{d}^{2n+4}x\;\sqrt{g}\;\Psi^\dagger\,\hat{\not{D}}\Psi\;. \tag{5.29}$$

Let us evaluate (5.29) on symmetric spinors given by

$$\Psi \;=\; \bigoplus_{i=0}^{m_1}\,\bigoplus_{\alpha=0}^{m_2}\Psi_{i\alpha} \qquad \text{with} \qquad \Psi_{i\alpha} \;=\; \begin{pmatrix}\psi^{(1)\,+}_{(m_1-2i)}\\ \psi^{(1)\,-}_{(m_1-2i)}\end{pmatrix}\otimes\begin{pmatrix}\psi^{(2)\,+}_{(m_2-2\alpha)}\\ \psi^{(2)\,-}_{(m_2-2\alpha)}\end{pmatrix} \tag{5.30}$$

with respect to the decomposition (5.20), where $\psi^{(\ell)\,\pm}_{(p)}$ are sections of $\mathcal{L}^{p\pm1}$ and $\Psi_{i\alpha}$ takes values in $\underline{\Delta}_{2n}\otimes\underline{V}_{k_{i\alpha}}$ with coefficient functions on M. After integration over $\mathbb{C}P^1\times\mathbb{C}P^1$, one easily computes analogously to [18] that the action functional (5.29) on symmetric spinors becomes

$$\begin{aligned}
S_{\mathrm{D}} \;&=\; 16\pi^2\,R_1^2\,R_2^2\,\int_M \mathrm{d}^{2n}x \qquad\qquad (5.31)\\
&\times\Bigg[\sum_{i=0}^{m_1^-}\sum_{\alpha=0}^{m_2^-}\sum_{k_1=0}^{m_1-2i-1}\sum_{k_2=0}^{m_2-2\alpha-1}\big(\psi^{(1)\,-}_{(m_1-2i)\,k_1}\,\psi^{(2)\,-}_{(m_2-2\alpha)\,k_2}\big)^\dagger\,\not{D}\big(\psi^{(1)\,-}_{(m_1-2i)\,k_1}\,\psi^{(2)\,-}_{(m_2-2\alpha)\,k_2}\big)\\
&+\sum_{i=0}^{m_1^-}\sum_{\alpha=m_2^+}^{m_2}\sum_{k_1=0}^{m_1-2i-1}\sum_{k_2=0}^{|m_2-2\alpha|-1}\big(\psi^{(1)\,-}_{(m_1-2i)\,k_1}\,\psi^{(2)\,+}_{(m_2-2\alpha)\,k_2}\big)^\dagger\,\not{D}\big(\psi^{(1)\,-}_{(m_1-2i)\,k_1}\,\psi^{(2)\,+}_{(m_2-2\alpha)\,k_2}\big)\\
&+\sum_{i=m_1^+}^{m_1}\sum_{\alpha=0}^{m_2^-}\sum_{k_1=0}^{|m_1-2i|-1}\sum_{k_2=0}^{m_2-2\alpha-1}\big(\psi^{(1)\,+}_{(m_1-2i)\,k_1}\,\psi^{(2)\,-}_{(m_2-2\alpha)\,k_2}\big)^\dagger\,\not{D}\big(\psi^{(1)\,+}_{(m_1-2i)\,k_1}\,\psi^{(2)\,-}_{(m_2-2\alpha)\,k_2}\big)\\
&+\sum_{i=m_1^+}^{m_1}\sum_{\alpha=m_2^+}^{m_2}\sum_{k_1=0}^{|m_1-2i|-1}\sum_{k_2=0}^{|m_2-2\alpha|-1}\big(\psi^{(1)\,+}_{(m_1-2i)\,k_1}\,\psi^{(2)\,+}_{(m_2-2\alpha)\,k_2}\big)^\dagger\,\not{D}\big(\psi^{(1)\,+}_{(m_1-2i)\,k_1}\,\psi^{(2)\,+}_{(m_2-2\alpha)\,k_2}\big)\Bigg],
\end{aligned}$$

where $\not{D} := \gamma^\mu\,D_\mu$ and the component functions $\psi^{(\ell)\,\pm}_{(p)\,k}(x)$ on M with $k=0,1,\dots,|p|-1$ form the irreducible representation $\underline{V}_{|p|}\cong\mathbb{C}^{|p|}$ of the group SU(2). The action functional (5.31) corresponds to a K-homology class $[\not{D}]$ in $\mathrm{K}^{\mathrm{a}}(M)$ twisted by appropriate monopole contributions and SU(2)×SU(2)-modules. We shall now proceed to describe this class more precisely.

5.3 K-theory charges

Consider a holomorphic chain as in (3.24) and suppose that it is a complex at the same time. Let us set $E_+ = \bigoplus_{i\,\text{even}} E_{ki0}$ and $E_- = \bigoplus_{i\,\text{odd}} E_{ki0}$, and define

$$\Phi \;:=\; \Big[\boldsymbol{\phi}^{(1)}_{(m,0)} + \big(\boldsymbol{\phi}^{(1)}_{(m,0)}\big)^\dagger\Big]\Big|_{E_-} \quad . \tag{5.32}$$

With respect to this grading, the graded connection (5.32) is an odd map $\Phi : E_- \to E_+$. Hence, the triple $\big[E_-, E_+; \Phi\big]$ represents the K-theory class of a brane-antibrane system with tachyon field Φ [34]. The same construction would carry through for a higher-rank quiver bundle of the form (2.12) if the latter was also a bi-complex, i.e. if both the horizontal and vertical arrows defined complexes. In this case the commutativity conditions (3.11) and (4.44) would allow us to lexicographically map the lattice onto a chain, and hence make contact with the above well-known K-theory construction.

However, for generic monopole numbers m_1 and m_2 the quiver bundle (2.12) does not have the requisite feature of a bi-complex due to the nilpotency properties (3.10). Following the interpretation of section 5.1 above, we need to fold the holomorphic lattice into maps between branes and antibranes [18, 34]. This accomplished by decomposing the quiver module (2.6) with respect to the multiplicative $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading defined by the signs of the monopole charges m_1-2i and $m_2-2\alpha$ at each vertex of $\mathsf{Q}_{(m_1,m_2)}$. As a $\mathbb{Z}_2$-graded vector space we have

$$\underline{\mathcal{V}} \;=\; \underline{\mathcal{V}}_+ \oplus \underline{\mathcal{V}}_- \qquad \text{with} \qquad \underline{\mathcal{V}}_+ \;=\; \underline{\mathcal{V}}_{++} \oplus \underline{\mathcal{V}}_{--} \quad \text{and} \quad \underline{\mathcal{V}}_- \;=\; \underline{\mathcal{V}}_{-+} \oplus \underline{\mathcal{V}}_{+-}\,, \tag{5.33}$$

where the bi-graded components are given analogously to (5.26) as

$$\begin{aligned}
\underline{\mathcal{V}}_{++} \;&=\; \bigoplus_{i=0}^{m_1^-} \bigoplus_{\alpha=0}^{m_2^-} \underline{V}_{ki\alpha} \qquad \text{and} \qquad \underline{\mathcal{V}}_{--} \;=\; \bigoplus_{i=m_1^+}^{m_1} \bigoplus_{\alpha=m_2^+}^{m_2} \underline{V}_{ki\alpha}\,, \\
\underline{\mathcal{V}}_{+-} \;&=\; \bigoplus_{i=0}^{m_1^-} \bigoplus_{\alpha=m_2^+}^{m_2} \underline{V}_{ki\alpha} \qquad \text{and} \qquad \underline{\mathcal{V}}_{-+} \;=\; \bigoplus_{i=m_1^+}^{m_1} \bigoplus_{\alpha=0}^{m_2^-} \underline{V}_{ki\alpha}\,.
\end{aligned} \tag{5.34}$$

Using (3.6)–(3.11), we now introduce the operators

$$\boldsymbol{\mu}^{(1)}_{(m_1,m_2)} \;:=\; \Big(\boldsymbol{\phi}^{(1)}_{(m_1,m_2)}\Big)^{m_1^-} \qquad \text{and} \qquad \boldsymbol{\mu}^{(2)}_{(m_1,m_2)} \;:=\; \Big(\boldsymbol{\phi}^{(2)}_{(m_1,m_2)}\Big)^{m_2^-} \tag{5.35}$$

constructed from the finite-energy Yang-Mills solutions of section 4.3. With respect to the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading in (5.33), they are odd maps

$$\begin{aligned}
\boldsymbol{\mu}^{(1)}_{(m_1,m_2)} &: \underline{\mathcal{V}}_{-\bullet} \otimes \mathcal{H} \longrightarrow \underline{\mathcal{V}}_{+\bullet} \otimes \mathcal{H} \qquad \text{with} \qquad \Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)}\Big)^2 \;=\; 0\,, \\
\boldsymbol{\mu}^{(2)}_{(m_1,m_2)} &: \underline{\mathcal{V}}_{\bullet-} \otimes \mathcal{H} \longrightarrow \underline{\mathcal{V}}_{\bullet+} \otimes \mathcal{H} \qquad \text{with} \qquad \Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)}\Big)^2 \;=\; 0
\end{aligned} \tag{5.36}$$

which together form the requisite bi-complex of noncommutative tachyon fields between branes and antibranes.

Let $\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}$ and $\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}$ denote the restrictions of the operators (5.35) to the isotopical component $\underline{V}_{k_{i\alpha}}$. These operators can be written in terms of bundle morphisms as

$$\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha} \;=\; \phi^{(1)}_{i-m_1^-\,\alpha}\cdots\phi^{(1)}_{i\alpha} \qquad \text{and} \qquad \boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha} \;=\; \phi^{(2)}_{i\,\alpha-m_2^-}\cdots\phi^{(2)}_{i\alpha}\,, \tag{5.37}$$

where it is understood that $\phi^{(1)}_{i\alpha} = 0 = \phi^{(2)}_{i\alpha}$ if $i<0$ or $\alpha<0$. From (4.37) and (4.38) it follows that the pair of operators (5.37) are respectively proportional to the Toeplitz operators

$$T^{(1)}_{i\alpha} \;:=\; T_{N_{i-m_1^-\,-1\,\alpha}}\,T^\dagger_{N_{i\alpha}} \qquad \text{and} \qquad T^{(2)}_{i\alpha} \;:=\; T_{N_{i\,\alpha-m_2^--1}}\,T^\dagger_{N_{i\alpha}}\,. \tag{5.38}$$

The tachyon fields (5.35) are thus holomorphic maps between branes of equal and opposite magnetic charges,

$$\begin{aligned}
\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha} &: \underline{V}_{k_{i\alpha}}\otimes\mathcal{H} \longrightarrow \underline{V}_{k_{i-m_1^--1\,\alpha}}\otimes\mathcal{H}\,,\\
\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha} &: \underline{V}_{k_{i\alpha}}\otimes\mathcal{H} \longrightarrow \underline{V}_{k_{i\,\alpha-m_2^--1}}\otimes\mathcal{H}\,,
\end{aligned} \tag{5.39}$$

with the implicit understanding that $\underline{V}_{k_{i\alpha}} = \{0\}$ when $i<0$ or $\alpha<0$. Furthermore, from (4.42) it follows that when the operators (5.37) are non-vanishing their kernels and cokernels are the finite dimensional vector spaces given by

$$\begin{aligned}
\ker\Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}\Big) \;&=\; \mathrm{im}\,P_{N_{i\alpha}} \qquad \text{and} \qquad \ker\Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}\Big)^\dagger \;=\; \mathrm{im}\,P_{N_{i-m_1^--1\,\alpha}}\,,\\
\ker\Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}\Big) \;&=\; \mathrm{im}\,P_{N_{i\alpha}} \qquad \text{and} \qquad \ker\Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}\Big)^\dagger \;=\; \mathrm{im}\,P_{N_{i\,\alpha-m_2^--1}}
\end{aligned} \tag{5.40}$$

with $N_{i\alpha} := 0$ for $i<0$ or $\alpha<0$.

The operators $\boldsymbol{\mu}^{(1)}_{(m_1,m_2)}$ and $\boldsymbol{\mu}^{(2)}_{(m_1,m_2)}$ are $k\times k$ matrices whose sum can be written as

$$\boldsymbol{\mu}^{(1)}_{(m_1,m_2)} \oplus \boldsymbol{\mu}^{(2)}_{(m_1,m_2)} = \begin{pmatrix} 0 & \boldsymbol{\mu}^{(1)}_{(m_1,m_2)-+} & \boldsymbol{\mu}^{(2)}_{(m_1,m_2)+-} & 0\\ 0 & 0 & 0 & \boldsymbol{\mu}^{(2)}_{(m_1,m_2)--}\\ 0 & 0 & 0 & \boldsymbol{\mu}^{(1)}_{(m_1,m_2)--}\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.41}$$

on $\underline{\mathcal{V}}\otimes\mathcal{H}$ with $\underline{\mathcal{V}} = \underline{\mathcal{V}}_{++}\oplus\underline{\mathcal{V}}_{-+}\oplus\underline{\mathcal{V}}_{+-}\oplus\underline{\mathcal{V}}_{--}$, where $\boldsymbol{\mu}^{(1)}_{(m_1,m_2)-\pm} := \boldsymbol{\mu}^{(1)}_{(m_1,m_2)}|_{\underline{\mathcal{V}}_{-\pm}\otimes\mathcal{H}}$ and $\boldsymbol{\mu}^{(2)}_{(m_1,m_2)\pm-} := \boldsymbol{\mu}^{(2)}_{(m_1,m_2)}|_{\underline{\mathcal{V}}_{\pm-}\otimes\mathcal{H}}$. This matrix presentation corresponds to the bundle diagram

$$\begin{array}{ccc}
\underline{\mathcal{V}}_{-+}\otimes\mathcal{H} & \xrightarrow{\;\boldsymbol{\mu}^{(1)}_{(m_1,m_2)-+}\;} & \underline{\mathcal{V}}_{++}\otimes\mathcal{H}\\
{\scriptstyle\boldsymbol{\mu}^{(2)}_{(m_1,m_2)--}}\big\uparrow & & \big\uparrow{\scriptstyle\boldsymbol{\mu}^{(2)}_{(m_1,m_2)+-}}\\
\underline{\mathcal{V}}_{--}\otimes\mathcal{H} & \xrightarrow[\;\boldsymbol{\mu}^{(1)}_{(m_1,m_2)--}\;]{} & \underline{\mathcal{V}}_{+-}\otimes\mathcal{H}\,.
\end{array} \tag{5.42}$$

Via an appropriate change of basis of the Hilbert space $\underline{\mathcal{V}} \otimes \mathcal{H}$, from (5.42) it follows that the operator (5.41) can be rewritten as

$$\boldsymbol{T}_{(m_1,m_2)} := \begin{pmatrix} 0\,0 & \boldsymbol{\mu}^{(1)}_{(m_1,m_2)-+} & \boldsymbol{\mu}^{(2)}_{(m_1,m_2)+-} \\ 0\,0 & \big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)--}\big)^\dagger & \big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)--}\big)^\dagger \\ 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 \end{pmatrix} \tag{5.43}$$

on $\underline{\mathcal{V}} \otimes \mathcal{H}$ with $\underline{\mathcal{V}} = \underline{\mathcal{V}}_{++} \oplus \underline{\mathcal{V}}_{--} \oplus \underline{\mathcal{V}}_{-+} \oplus \underline{\mathcal{V}}_{+-}$.

The important ingredients in this construction are the holomorphic relations $\mathsf{R}_{(m_1,m_2)}$ of the quiver $\mathsf{Q}_{(m_1,m_2)}$ which enable us to commute the graded connections as in (3.11), along with the non-holomorphic relations (4.44). Together they imply that, with respect to the $\mathbb{Z}_2$-grading in (5.33), the operator (5.43) is an odd map

$$\boldsymbol{T}_{(m_1,m_2)} : \underline{\mathcal{V}}_- \otimes \mathcal{H} \longrightarrow \underline{\mathcal{V}}_+ \otimes \mathcal{H} \qquad \text{with} \qquad \big(\boldsymbol{T}_{(m_1,m_2)}\big)^2 = 0 \tag{5.44}$$

and hence it produces the appropriate two-term complex representing the brane-antibrane system with noncommutative tachyon field (5.43). Again, when acting on isotopical components the operator $\boldsymbol{T}_{(m_1,m_2)i\alpha}$ relates a given brane to the two possible antibranes of equal but opposite charge as

$$\begin{aligned} \boldsymbol{T}_{(m_1,m_2)i\alpha}\Big|_{\underline{\mathcal{V}}_{-+}} &: \underline{V}_{k_{i\alpha}} \otimes \mathcal{H} \longrightarrow \big(\underline{V}_{k_{i-m_1^- -1\,\alpha}} \otimes \mathcal{H}\big) \oplus \big(\underline{V}_{k_{i\,\alpha+m_2^-+1}} \otimes \mathcal{H}\big)\,, \\ \boldsymbol{T}_{(m_1,m_2)i\alpha}\Big|_{\underline{\mathcal{V}}_{+-}} &: \underline{V}_{k_{i\alpha}} \otimes \mathcal{H} \longrightarrow \big(\underline{V}_{k_{i\,\alpha-m_2^- -1}} \otimes \mathcal{H}\big) \oplus \big(\underline{V}_{k_{i+m_1^-+1\,\alpha}} \otimes \mathcal{H}\big)\,. \end{aligned} \tag{5.45}$$

From (5.40) it then follows that the operators (5.45) have kernels and cokernels of finite dimensions given by

$$\begin{aligned} \dim\ker\Big(\boldsymbol{T}_{(m_1,m_2)i\alpha}\Big)^\dagger\Big|_{\underline{\mathcal{V}}_{++}} &= \dim\Big[\ker\Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}\Big) \cap \ker\Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}\Big)^\dagger\Big] = N_{i-m_1^- -1\,\alpha-m_2^- -1}\,, \\ \dim\ker\Big(\boldsymbol{T}_{(m_1,m_2)i\alpha}\Big)^\dagger\Big|_{\underline{\mathcal{V}}_{--}} &= \dim\Big[\ker\Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}\Big)^\dagger \cap \ker\Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}\Big)\Big] = N_{i\alpha}\,, \\ \dim\ker\Big(\boldsymbol{T}_{(m_1,m_2)i\alpha}\Big)\Big|_{\underline{\mathcal{V}}_{-+}} &= \dim\Big[\ker\Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}\Big)^\dagger \cap \ker\Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}\Big)^\dagger\Big] = N_{i\,\alpha-m_2^- -1}\,, \\ \dim\ker\Big(\boldsymbol{T}_{(m_1,m_2)i\alpha}\Big)\Big|_{\underline{\mathcal{V}}_{+-}} &= \dim\Big[\ker\Big(\boldsymbol{\mu}^{(1)}_{(m_1,m_2)i\alpha}\Big) \cap \ker\Big(\boldsymbol{\mu}^{(2)}_{(m_1,m_2)i\alpha}\Big)\Big] = N_{i-m_1^- -1\,\alpha}\,. \end{aligned} \tag{5.46}$$

To incorporate the twistings by the magnetic monopole bundles, we use the ABS construction of section 5.2 above to modify the tachyon field (5.43) to the operator

$$\boldsymbol{\mathcal{T}}_{(m_1,m_2)} := \boldsymbol{T}_{(m_1,m_2)} \otimes \mathbf{1} : \underline{\Delta}^-_{\underline{\mathcal{V}}} \otimes \mathcal{H} \longrightarrow \underline{\Delta}^+_{\underline{\mathcal{V}}} \otimes \mathcal{H} \tag{5.47}$$

where $\underline{\Delta}^+_{\underline{\mathcal{V}}} := \underline{\Delta}^{++}_{\underline{\mathcal{V}}} \oplus \underline{\Delta}^{--}_{\underline{\mathcal{V}}}$ and $\underline{\Delta}^-_{\underline{\mathcal{V}}} := \underline{\Delta}^{-+}_{\underline{\mathcal{V}}} \oplus \underline{\Delta}^{+-}_{\underline{\mathcal{V}}}$. The corresponding tachyon operators (5.35) then define noncommutative versions of the Clifford multiplications (5.28).

JHEP09(2006)054

Since $\dim \underline{V}_{|p|} = |p|$, from (5.26), (5.27) and (5.46) it follows that the index of the tachyon field (5.47) is given by

$$\begin{aligned}\operatorname{index}\big(\boldsymbol{\mathcal{T}}_{(m_1,m_2)}\big) &= \dim\ker\big(\boldsymbol{\mathcal{T}}_{(m_1,m_2)}\big) - \dim\ker\big(\boldsymbol{\mathcal{T}}_{(m_1,m_2)}\big)^\dagger \\ &= \sum_{i=m_1^+}^{m_1}\sum_{\alpha=m_2^+}^{m_2} |m_1-2i|\,|m_2-2\alpha| \\ &\quad\times\Big[\big(N_{i\,\alpha-m_2^- -1}+N_{i-m_1^- -1\,\alpha}\big)-\big(N_{i-m_1^- -1\,\alpha-m_2^- -1}+N_{i\alpha}\big)\Big] \\ &= -Q\ . \end{aligned} \tag{5.48}$$

The virtual Euler class generated by the cohomology of the complex (5.44) is the analytic K-homology class in $\mathrm{K}^{\mathrm{a}}(\mathbb{R}^{2n})$ of the configuration of D-branes represented by the quiver bundle (2.12). The formula (5.48) then asserts that the K-theory charge of the noncommutative quiver vortex configuration constructed in section 4.3, i.e. the virtual dimension of this index class, coincides with the Yang-Mills instanton charge (5.5)–(5.7) on $\mathbb{R}^{2n}_\theta\times S^2\times S^2$. The corresponding geometric worldvolume description in terms of topological K-cycles may now also be worked out in exactly the same way as in [18]. It relies crucially on the equivariant excision theorem (5.9) which asserts the equivalence of the brane configurations on $M\times\mathbb{C}P^1\times\mathbb{C}P^1$ and on M.

5.4 D-brane categories

The K-theory construction in section 5.3 above of the brane configuration corresponding to the quiver bundle (2.12) is somewhat primitive in that it only builds the system at the level of topological charges. In particular, it relies crucially on the equivariant excision theorem (5.9). We can get a more detailed picture of the dynamics of these D-branes, and in particular how the original configuration folds itself into branes and antibranes, by modelling our instanton solutions in the category of quiver representations of $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$. The ensuing homological algebra of this category will then exemplify the roles of the SU(2)×SU(2)-modules and of the relations of the quiver in computing the equivariant charges. Our previous approach based on intersection pairings at the K-theory level misses certain quantitative aspects of the brane configurations corresponding to the quiver bundle (2.12), while the category of quiver representations provides a rigorous and complete framework for understanding these systems [10].

Let us fix a vertex $(m_1-2i,m_2-2\alpha)\in\mathsf{Q}^{(0)}_{(m_1,m_2)}$ of the quiver and consider the distinguished representations $\underline{\mathcal{P}}_{i\alpha}$ and $\underline{\mathcal{L}}_{i\alpha}$ introduced in sections 3.1 and 4.4 respectively. Then one has a canonical projective resolution given by the exact sequence [21]

$$0\ \longrightarrow\ \underline{\mathcal{P}}_{i-1\,\alpha-1}\ \longrightarrow\ \underline{\mathcal{P}}_{i-1\,\alpha}\oplus\underline{\mathcal{P}}_{i\,\alpha-1}\ \longrightarrow\ \underline{\mathcal{P}}_{i\alpha}\ \longrightarrow\ \underline{\mathcal{L}}_{i\alpha}\ \longrightarrow\ 0\ . \tag{5.49}$$

The first term corresponds to the independent relations of the quiver which are indexed by (i,α) with paths starting at (i,α) and ending at $(i-1,\alpha-1)$. The second sum corresponds to the arrows which start at node (i,α). Since there are no "relations among the relations", there are no further non-trivial modules to the far left of the exact sequence (5.49).

Consider now the module (4.46) generated by a fixed noncommutative instanton solution. From section 4.5 it follows that this quiver representation specifies the loci of the D-branes in $\mathbb{R}^{2n}$, and since all the moduli of our solutions come from the noncommutative quiver solitons it will suffice to recover the appropriate topological charge. Taking the tensor product of (5.49) with the components $\ker T^\dagger_{N_{i\alpha}}$ of $\underline{\mathcal{I}}$ and summing over all nodes (i,α) of the quiver $\mathsf{Q}_{(m_1,m_2)}$ gives the projective Ringel resolution of $\underline{\mathcal{I}}$ as

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \underline{\mathcal{P}}_{i-1\,\alpha-1} \otimes \ker T^\dagger_{N_{i\alpha}} \longrightarrow \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \big(\underline{\mathcal{P}}_{i-1\,\alpha} \oplus \underline{\mathcal{P}}_{i\,\alpha-1}\big) \otimes \ker T^\dagger_{N_{i\alpha}} \longrightarrow \\ \longrightarrow \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \underline{\mathcal{P}}_{i\alpha} \otimes \ker T^\dagger_{N_{i\alpha}} \longrightarrow \underline{\mathcal{I}} \longrightarrow 0 . \end{aligned} \tag{5.50}$$

Let

$$\underline{\mathcal{W}} = \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \underline{W}_{i\alpha} \qquad \text{with} \qquad \vec{k}_{\underline{\mathcal{W}}} = \big(w_{i\alpha}\big)^{i=0,1,\ldots,m_1}_{\alpha=0,1,\ldots,m_2} \tag{5.51}$$

be any other representation of $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$. It will be fixed below to correctly incorporate the monopole fields at the vertices of the quiver. Applying the contravariant functor $\mathrm{Hom}(-\,,\,\underline{\mathcal{W}})$ to the projective resolution (5.50) using (3.2) then induces the complex

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) \longrightarrow \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i\alpha}\big) \longrightarrow \\ \longrightarrow \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \Big(\mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i-1\,\alpha}\big) \oplus \mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i\,\alpha-1}\big)\Big) \longrightarrow \\ \longrightarrow \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i-1\,\alpha-1}\big) \longrightarrow \mathrm{Ext}^2\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) \longrightarrow 0 . \end{aligned} \tag{5.52}$$

The group $\mathrm{Ext}^p(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}})$ is defined to be the cohomology of the complex (5.52) in the p-th position. One has $\mathrm{Ext}^0(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}) = \mathrm{Hom}(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}})$ corresponding to the vertices of the quiver $\mathsf{Q}_{(m_1,m_2)}$. This group classifies morphisms $\underline{f} : \underline{\mathcal{I}} \to \underline{\mathcal{W}}$ of quiver representations as in section 3.1 and represents the partial gauge symmetries of the combined system of D-branes and magnetic monopoles. The group $\mathrm{Ext}^1(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}) = \mathrm{Ext}(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}})$ corresponds to the arrows of the quiver and classifies the $\mathsf{Q}_{(m_1,m_2)}$-modules $\underline{\mathcal{U}}$ which can be defined by short exact sequences

$$0 \longrightarrow \underline{\mathcal{I}} \xrightarrow{\underline{f}} \underline{\mathcal{U}} \xrightarrow{\underline{g}} \underline{\mathcal{W}} \longrightarrow 0 . \tag{5.53}$$

We may regard the module $\underline{\mathcal{U}}$ as a deformation of $\underline{\mathcal{I}} \oplus \underline{\mathcal{W}}$ which simulates the attaching of magnetic monopoles to the D-branes to form a bound state $\underline{\mathcal{U}}$. The arrows of (5.53) are given by morphisms $\underline{f} \in \mathrm{Hom}(\underline{\mathcal{I}}\,,\,\underline{\mathcal{U}})$ and $\underline{g} \in \mathrm{Hom}(\underline{\mathcal{U}}\,,\,\underline{\mathcal{W}})$, reflecting the fact that $\underline{\mathcal{I}}$ and $\underline{\mathcal{W}}$ are constituents of $\underline{\mathcal{U}}$ arising from partial gauge symmetries. Finally, the non-trivial Ext^2 group accounts for the relations $\mathsf{R}_{(m_1,m_2)}$, while $\mathrm{Ext}^p = 0$ for all $p \geq 3$ since there are no relations among our relations.

We now define the charge of the given configuration of noncommutative instantons relative to the $(\mathsf{Q}_{(m_1,m_2)}\,,\,\mathsf{R}_{(m_1,m_2)})$-module (5.51) through the relative Euler character

$$\chi\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) := \sum_{p\geq 0}(-1)^p\ \dim\,\mathrm{Ext}^p\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big)\ . \tag{5.54}$$

This coincides with the Ringel form on the representation ring $\mathrm{R}_{\mathsf{A}_{(m_1,m_2)}}$ of the quiver $\mathsf{Q}_{(m_1,m_2)}$. Using (5.52) we may compute the Euler form as

$$\begin{aligned}\chi\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) &= \dim\,\mathrm{Hom}\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) + \dim\,\mathrm{Ext}^2\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) - \dim\,\mathrm{Ext}\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big)\\ &= \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2}\dim\,\mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i\alpha}\big) + \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2}\dim\,\mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i-1\,\alpha-1}\big)\\ &\quad - \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2}\Big(\dim\,\mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i-1\,\alpha}\big) + \dim\,\mathrm{Hom}\big(\ker T^\dagger_{N_{i\alpha}}\,,\,\underline{W}_{i\,\alpha-1}\big)\Big)\\ &= \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2}N_{i\alpha}\left(w_{i\alpha}+w_{i-1\,\alpha-1}-w_{i-1\,\alpha}-w_{i\,\alpha-1}\right)\ .\end{aligned} \tag{5.55}$$

Following [18], we choose the coupling representation (5.51) to the brane configuration of the quiver bundle (2.12) to correctly incorporate the magnetic monopole charges through the appropriate folding of SU(2)×SU(2)-representations appearing in the ABS construction (5.25)–(5.27). We define a non-decreasing sequence $\underline{W}_{i\alpha}\subseteq\underline{W}_{j\beta}$, $i\leq j,\alpha\leq\beta$ of representations as we move along the quiver of constituent D-branes such that the SU(2)×SU(2)-module $\underline{W}_{i\alpha}$ gives an extension of the monopole fields carried by the elementary brane state at node (i,α). Thus we take

$$\underline{W}_{i\alpha} = \bigoplus_{j=0}^{i-1}\bigoplus_{\beta=0}^{\alpha-1}\underline{V}_{|m_1-2j|}\otimes\underline{V}_{|m_2-2\beta|}\ . \tag{5.56}$$

As an element of the representation ring $\mathrm{R}_{\mathsf{A}_{(m_1,m_2)}}$ of the quiver $\mathsf{Q}_{(m_1,m_2)}$, we view the module (5.56) as a graded sum of representations with respect to the signs of the monopole charges such that its virtual dimension is given by

$$\begin{aligned}w_{i\alpha} &= \dim\big[\underline{W}_{i\alpha}\big]^{\mathrm{vir}}\\ &= \sum_{j=0}^{i-1}\sum_{\beta=0}^{\alpha-1}(m_1-2j)\,(m_2-2\beta) \ =\ i\,\alpha\,(m_1-i+1)\,(m_2-\alpha+1)\ .\end{aligned} \tag{5.57}$$

One easily checks that the integers (5.57) obey the inhomogeneous recursion relation

$$w_{i\alpha}+w_{i-1\,\alpha-1}-w_{i-1\,\alpha}-w_{i\,\alpha-1} = (m_1-2i)\,(m_2-2\alpha)\ . \tag{5.58}$$

Consequently, the Euler-Ringel form (5.55) in this case becomes

$$\chi\big(\underline{\mathcal{I}}\,,\,\underline{\mathcal{W}}\big) \ =\ \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2}N_{i\alpha}\,(m_1-2i)\,(m_2-2\alpha) \ =\ Q\ , \tag{5.59}$$

reproducing again the instanton charge (5.5). The equivalence between the Euler characteristic (5.54) and the K-theory charge of section 5.3 above is a consequence of the index theorem applied to the complex generating the cohomology groups $\mathrm{H}^p(\mathbb{R}^{2n}_\theta,\underline{\mathcal{I}}\otimes\underline{\mathcal{W}}^\vee\otimes\mathcal{H})$.

Acknowledgments

This work was supported in part by the Deutsche Forschungsgemeinschaft (DFG). The work of R.J.S. was supported in part by PPARC Grant PPA/G/S/2002/00478 and by the EU-RTN Network Grant MRTN-CT-2004-005104.

References

[1] A. Sen, *Tachyon dynamics in open string theory*, *Int. J. Mod. Phys.* **A 20** (2005) 5513 [hep-th/0410103].

[2] M. Alishahiha, H. Ita and Y. Oz, *On superconnections and the tachyon effective action*, *Phys. Lett.* **B 503** (2001) 181 [hep-th/0012222];
R.J. Szabo, *Superconnections, anomalies and non-BPS brane charges*, *J. Geom. Phys.* **43** (2002) 241 [hep-th/0108043].

[3] M.R. Douglas, *Branes within branes*, hep-th/9512077.

[4] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *JHEP* **06** (2000) 022 [hep-th/0005006];
J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as non-commutative solitons*, *JHEP* **07** (2000) 042 [hep-th/0005031];
M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *JHEP* **04** (2001) 001 [hep-th/0009142];
J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *JHEP* **12** (2000) 024 [hep-th/0010060];
D.J. Gross and N.A. Nekrasov, *Solitons in noncommutative gauge theory*, *JHEP* **03** (2001) 044 [hep-th/0010090].

[5] Y. Matsuo, *Topological charges of noncommutative soliton*, *Phys. Lett.* **B 499** (2001) 223 [hep-th/0009002];
J.A. Harvey and G.W. Moore, *Noncommutative tachyons and K-theory*, *J. Math. Phys.* **42** (2001) 2765 [hep-th/0009030].

[6] R. Minasian and G.W. Moore, *K-theory and Ramond-Ramond charge*, *JHEP* **11** (1997) 002 [hep-th/9710230];
E. Witten, *D-branes and K-theory*, *JHEP* **12** (1998) 019 [hep-th/9810188];
P. Hořava, *Type IIA D-branes, K-theory and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1999) 1373 [hep-th/9812135];
K. Olsen and R.J. Szabo, *Constructing D-branes from K-theory*, *Adv. Theor. Math. Phys.* **3** (1999) 889 [hep-th/9907140];
E. Witten, *Overview of K-theory applied to strings*, *Int. J. Mod. Phys.* **A 16** (2001) 693 [hep-th/0007175];
T. Asakawa, S. Sugimoto and S. Terashima, *D-branes, matrix theory and K-homology*, *JHEP* **03** (2002) 034 [hep-th/0108085];
R.J. Szabo, *D-branes, tachyons and K-homology*, *Mod. Phys. Lett.* **A 17** (2002) 2297 [hep-th/0209210];
J.J. Manjarín, *Topics on D-brane charges with B-fields*, *Int. J. Geom. Meth. Mod. Phys.* **1** (2004) 545 [hep-th/0405074].

[7] J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, hep-th/0102076;

M. Hamanaka, *Noncommutative solitons and D-branes*, hep-th/0303256;
R.J. Szabo, *D-branes in noncommutative field theory*, hep-th/0512054.

[8] D.J. Gross and N.A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, *JHEP* **10** (2000) 021 [hep-th/0007204];
L. Hadasz, U. Lindström, M. Roček and R. von Unge, *Noncommutative multisolitons: moduli spaces, quantization, finite theta effects and stability*, *JHEP* **06** (2001) 040 [hep-th/0104017];
O. Lechtenfeld and A.D. Popov, *Noncommutative multi-solitons in 2+1 dimensions*, *JHEP* **11** (2001) 040 [hep-th/0106213]; *Scattering of noncommutative solitons in 2+1 dimensions*, *Phys. Lett.* **B 523** (2001) 178 [hep-th/0108118];
M. Hamanaka, Y. Imaizumi and N. Ohta, *Moduli space and scattering of D0-branes in noncommutative super Yang-Mills theory*, *Phys. Lett.* **B 529** (2002) 163 [hep-th/0112050];
K. Furuta, T. Inami, H. Nakajima and M. Yamamoto, *Low-energy dynamics of noncommutative CP(1) solitons in 2+1 dimensions*, *Phys. Lett.* **B 537** (2002) 165 [hep-th/0203125];
M. Wolf, *Soliton-antisoliton scattering configurations in a noncommutative sigma model in 2+1 dimensions*, *JHEP* **06** (2002) 055 [hep-th/0204185];
M. Ihl and S. Uhlmann, *Noncommutative extended waves and soliton-like configurations in* $N = 2$ *string theory*, *Int. J. Mod. Phys.* **A 18** (2003) 4889 [hep-th/0211263].

[9] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167;
C.V. Johnson and R.C. Myers, *Aspects of type IIB theory on ALE spaces*, *Phys. Rev.* **D 55** (1997) 6382 [hep-th/9610140];
M.R. Douglas, B. Fiol and C. Romelsberger, *The spectrum of BPS branes on a noncompact Calabi-Yau*, *JHEP* **09** (2005) 057 [hep-th/0003263].

[10] M.R. Douglas, *D-branes, categories and* $N = 1$ *supersymmetry*, *J. Math. Phys.* **42** (2001) 2818 [hep-th/0011017];
D. Berenstein and M.R. Douglas, *Seiberg duality for quiver gauge theories*, hep-th/0207027;
P.S. Aspinwall and I.V. Melnikov, *D-branes on vanishing del Pezzo surfaces*, *JHEP* **12** (2004) 042 [hep-th/0405134].

[11] S.K. Donaldson, *Anti-self-dual Yang-Mills connections on a complex algebraic surface and stable vector bundles*, *Proc. Lond. Math. Soc.* **50** (1985) 1; *Infinite determinants, stable bundles and curvature*, *Duke Math. J.* **54** (1987) 231;
K.K. Uhlenbeck and S.-T. Yau, *On the existence of hermitian Yang-Mills connections on stable bundles over compact Kähler manifolds*, *Commun. Pure Appl. Math.* **39** (1986) 257; *A note on our previous paper*, *ibid.* **42** (1989) 703.

[12] O. García-Prada, *Invariant connections and vortices*, *Commun. Math. Phys.* **156** (1993) 527; *Dimensional reduction of stable bundles, vortices and stable pairs*, *Int. J. Math.* **5** (1994) 1.

[13] L. Álvarez-Cónsul and O. García-Prada, *Dimensional reduction,* $\mathrm{SL}(2,\mathbb{C})$*-equivariant bundles and stable holomorphic chains*, *Int. J. Math.* **12** (2001) 159

[14] L. Álvarez-Cónsul and O. García-Prada, *Dimensional reduction and quiver bundles*, *J. Reine Angew. Math.* **556** (2003) 1 [math.DG/0112160].

[15] L. Alvarez-Cónsul and O. García-Prada, *Hitchin-Kobayashi correspondence, quivers and vortices*, *Commun. Math. Phys.* **238** (2003) 1 [math.dg/0112161].

[16] T.A. Ivanova and O. Lechtenfeld, *Noncommutative multi-instantons on* $R^{2n} \times S^2$, *Phys. Lett.* **B 567** (2003) 107 [hep-th/0305195].

JHEP09(2006)054

[17] O. Lechtenfeld, A.D. Popov and R.J. Szabo, *Noncommutative instantons in higher dimensions, vortices and topological K-cycles*, *JHEP* **12** (2003) 022 [hep-th/0310267].

[18] A.D. Popov and R.J. Szabo, *Quiver gauge theory of nonabelian vortices and noncommutative instantons in higher dimensions*, *J. Math. Phys.* **47** (2006) 012306 [hep-th/0504025].

[19] A. Konechny and A.S. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, *Phys. Rept.* **360** (2002) 353 [hep-th/0012145,hep-th/0107251];
M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048];
R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[20] P.B. Gothen and A.D. King, *Homological algebra of twisted quiver bundles*, *J. London Math. Soc.* **71** (2005) 85 [math.AG/0202033].

[21] M. Auslander, I. Reiten and S.O. Smalø, *Representation theory of Artin algebras*, Cambridge University Press, 1995;
D.J. Benson, *Representations and cohomology*, Cambridge University Press, 1998

[22] N. Nekrasov and A.S. Schwarz, *Instantons on noncommutative R^4 and (2,0) superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [hep-th/9802068];
N.A. Nekrasov, *Noncommutative instantons revisited*, *Commun. Math. Phys.* **241** (2003) 143 [hep-th/0010017];
C.-S. Chu, V.V. Khoze and G. Travaglini, *Notes on noncommutative instantons*, *Nucl. Phys.* **B 621** (2002) 101 [hep-th/0108007];
M. Hamanaka, *ADHM/NAHM construction of localized solitons in noncommutative gauge theories*, *Phys. Rev.* **D 65** (2002) 085022 [hep-th/0109070].

[23] O. Lechtenfeld and A.D. Popov, *Noncommutative 't Hooft instantons*, *JHEP* **03** (2002) 040 [hep-th/0109209];
Y. Tian and C.-J. Zhu, *Comments on noncommutative ADHM construction*, *Phys. Rev.* **D 67** (2003) 045016 [hep-th/0210163];
Z. Horváth, O. Lechtenfeld and M. Wolf, *Noncommutative instantons via dressing and splitting approaches*, *JHEP* **12** (2002) 060 [hep-th/0211041];
T.A. Ivanova, O. Lechtenfeld and H. Müller-Ebhardt, *Noncommutative moduli for multi-instantons*, *Mod. Phys. Lett.* **A 19** (2004) 2419 [hep-th/0404127].

[24] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063];
R. Wimmer, *D0-D4 brane tachyon condensation to a BPS state and its excitation spectrum in noncommutative super Yang-Mills theory*, *JHEP* **05** (2005) 022 [hep-th/0502158];
M. Billo, M. Frau, S. Sciuto, G. Vallone and A. Lerda, *Non-commutative (D)-instantons*, *JHEP* **05** (2006) 069 [hep-th/0511036];
O. Lechtenfeld and C. Sämann, *Matrix models and D-branes in twistor string theory*, *JHEP* **03** (2006) 002 [hep-th/0511130].

[25] E. Witten, *Monopoles and four manifolds*, *Math. Res. Lett.* **1** (1994) 769 [hep-th/9411102].

[26] A.D. Popov, A.G. Sergeev and M. Wolf, *Seiberg-Witten monopole equations on noncommutative R^4*, *J. Math. Phys.* **44** (2003) 4527 [hep-th/0304263];
A. Sako and T. Suzuki, *Dimensional reduction of Seiberg-Witten monopole equations, $N = 2$ noncommutative supersymmetric field theory and Young diagram*, hep-th/0511085.

JHEP09(2006)054

[27] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043];
D.P. Jatkar, G. Mandal and S.R. Wadia, *Nielsen-Olesen vortices in noncommutative Abelian Higgs model*, *JHEP* **09** (2000) 018 [hep-th/0007078];
D. Bak, *Exact multi-vortex solutions in noncommutative Abelian Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204];
D. Bak, K.-M. Lee and J.-H. Park, *Noncommutative vortex solitons*, *Phys. Rev.* **D 63** (2001) 125010 [hep-th/0011099].

[28] V.G. Kac, *Infinite root systems, representations of graphs and invariant theory*, *Invent. Math.* **56** (1980) 57.

[29] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Springer-Verlag, Berlin, 1994.

[30] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, *Commun. Math. Phys.* **233** (2003) 355 [hep-th/0103256];
T. Amdeberhan and A. Ayyer, *Towards the moduli space of extended partial isometries*, hep-th/0508014.

[31] Yu.A. Drozd and V.V. Kirichenko, *Finite dimensional algebras*, Springer, Berlin, 1994;
C.A. Weibel, *Introduction to homological algebra*, Cambridge Studies Adv. Math. **38**, Cambridge University Press, 1995.

[32] G.B. Segal, *The representation ring of a compact Lie group*, *Publ. Math. IHES (Paris)* **34** (1968) 113; *Equivariant K-theory*, *Publ. Math. IHES (Paris)* **34** (1968) 129.

[33] M.F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, *Topology* **3** (1964) 3.

[34] E.R. Sharpe, *D-branes, derived categories and Grothendieck groups*, *Nucl. Phys.* **B 561** (1999) 433 [hep-th/9902116];
Y. Oz, T. Pantev and D. Waldram, *Brane-antibrane systems on Calabi-Yau spaces*, *JHEP* **02** (2001) 045 [hep-th/0009112].

JHEP09(2006)054